Shape optimization for contact problem involving Signorini unilateral conditions

Aymeric Jacob de Cordemoy

Shape optimization for contact problem involving Signorini unilateral conditions

Aymeric Jacob de Cordemoy*

March 26, 2023

Abstract

This paper investigates a shape optimization problem involving the Signorini unilateral conditions in a linear elastic model, without any penalization procedure. The shape sensitivity analysis is performed using tools from convex and variational analysis such as proximal operators and the notion of twice epi-differentiability. We prove that the solution to the Signorini problem admits a directional derivative with respect to the shape, and we characterize it as the solution to another Signorini problem. Then, the shape gradient of the corresponding energy functional is explicitly characterized which allows us to perform numerical simulations to illustrate this methodology.

Keywords: Shape optimization, shape sensitivity analysis, shape derivative, Signorini unilateral conditions, variational inequalities, proximal operator, twice epi-differentiability.

AMS Classification: 49Q10, 49J40, 35J86, 74M15, 74P10.

1 Introduction

Motivation. On the one hand, mechanical contact models are used to study the deformation of solids that touch each other on parts of their boundaries. One of the mechanical setting consists in a deformable body which is in contact with a rigid foundation without penetrating it and frictionless. From the mathematical point of view, the non-permeability conditions take the form of inequalities on the contact surface called *Signorini unilateral conditions* (see, e.g., [37, 38]). Thus, those mechanical contact problems are usually investigated through the theory of variational inequalities, and the Signorini unilateral conditions cause nonlinearities in the corresponding variational formulations. On the other hand, shape optimization is the mathematical field aimed at finding the optimal shape of a given object for a given criterion, that is the shape which minimizes a certain cost functional while satisfying given constraints. In order to numerically solve a shape optimization problem, the standard gradient descent method requires to compute the *shape gradient* of the cost functional.

Shape optimization problems with mechanical contact models involving Signorini unilateral conditions have already been studied in the literature, and classical techniques to compute *material* and *shape derivatives* are based on Mignot's theorem (see [26]) about the *conical differentiability* of projection operators on nonempty *polyhedric* closed convex sets (see, e.g, [16, 25, 39]). The material and shape derivatives are usually characterized with abstract variational inequalities, thus cause difficulties to compute a suitable shape gradient of the cost functional. These difficulties are

*Université de Pau et des Pays de l'Adour, E2S UPPA, CNRS, LMAP, UMR 5142, 64000 Pau, France. aymeric.jacob-de-cordemoy@univ-pau.fr

usually solved in the literature using a penalization procedure (see, e.g., [22]), which consists in adding a penalty functional in the optimization problem associated with the model, in order to handle the constraints due to the Signorini unilateral conditions (for instance by considering the Moreau's envelope of the indicator function of the constraint set). Hence, the optimality condition is described by a variational equality (see, e.g., [6, 9, 21, 23]). However this penalization method does not take into account the exact characterization of the solution and may perturb the original nature of the model.

In this paper we investigate a shape optimization problem involving the Signorini unilateral conditions, using a new methodology based on tools from convex and variational analysis such as the notion of *proximal operator* introduced by J.J. Moreau in 1965 (see [28]) and the notion of *twice epi-differentiability* introduced by R.T. Rockafellar in 1985 (see [31]). Note that we have already studied the feasibility of this methodology on a shape optimization problem involving the *scalar Tresca friction law* (see [3]). First this new methodology allows us to recover the same results obtained in [9], [25, Chapter 5 Section 5.2 p.111] and [39, Chapter 4 Section 4.6 p.205]. Indeed, if a nonempty closed convex set is polyhedric, then from Mignot's theorem the projection operator on this set is conically differentiable, and its conical derivative coincides with the proximal operator associated with the *second-order epi-derivative* of the appropriate indicator function, and thus our approach coincides with that used in the literature. Second the main novelty of the present work is that, under appropriate assumptions, our method permits to characterize the material and shape derivatives of the solution to the Signorini problem as the solutions to other Signorini problems. This point, to the best of our knowledge, has never been noticed in the literature. Furthermore, by using this new characterization, we obtain an explicit expression of the shape gradient of the corresponding energy functional. Therefore, without using any penalization procedure, the present work can be seen as a complement and an extension of the previous articles on this subject.

Description of the shape optimization problem and methodology. In this paragraph, we use standard notations which are recalled in Subsection 2.4. Let $d \in \{2,3\}$ which represents the dimension, f be a function in $\mathrm{H}^1(\mathbb{R}^d,\mathbb{R}^d)$, Ω_{ref} be a nonempty connected bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary $\Gamma_{\mathrm{ref}} := \partial\Omega_{\mathrm{ref}}$, such that $\Gamma_{\mathrm{ref}} = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}_{\mathrm{ref}}}$, where Γ_{D} and $\Gamma_{\mathrm{S}_{\mathrm{ref}}}$ are two measurable pairwise disjoint subsets of Γ_{ref}, and Γ_{D} has a positive measure.

In this paper we consider the shape optimization problem given by

$$\underset{\substack{\Omega\in\mathcal{U}_{\mathrm{ref}}\\ |\Omega|=|\Omega_{\mathrm{ref}}|}}{\text{minimize}} \ \mathcal{J}(\Omega), \tag{1.1}$$

where

$$\mathcal{U}_{\mathrm{ref}} := \Big\{\Omega \subset \mathbb{R}^d \mid \Omega \text{ nonempty connected bounded open subset of } \mathbb{R}^d$$
$$\text{with Lipschitz boundary } \Gamma := \partial\Omega \text{ such that } \Gamma_{\mathrm{D}} \subset \Gamma\Big\}, \tag{1.2}$$

with the volume constraint $|\Omega| = |\Omega_{\mathrm{ref}}| > 0$, Ω is an elastic solid satisfying the linear elastic model, for all $\Omega \in \mathcal{U}_{\mathrm{ref}}$, and where $\mathcal{J} : \mathcal{U}_{\mathrm{ref}} \to \mathbb{R}$ is the *Signorini energy functional* defined by

$$\mathcal{J}(\Omega) := \frac{1}{2}\int_{\Omega} \mathrm{Ae}\,(u_\Omega) : \mathrm{e}\,(u_\Omega) - \int_{\Omega} f \cdot u_\Omega, \tag{1.3}$$

where $u_\Omega \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ stands for the unique solution to the Signorini problem given by

$$\left\{\begin{aligned} -\mathrm{div}(\mathrm{Ae}(u)) &= f && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma_{\mathrm{D}},\\ \sigma_\tau(u) &= 0 && \text{on } \Gamma_{\mathrm{S}},\\ u_{\mathrm{n}} \leq 0, \sigma_{\mathrm{n}}(u) \leq 0 \text{ and } u_{\mathrm{n}}\sigma_{\mathrm{n}}(u) &= 0 && \text{on } \Gamma_{\mathrm{S}}, \end{aligned}\right. \tag{SP$_\Omega$}$$

where, for all $\Omega \in \mathcal{U}_{\mathrm{ref}}$, $\Gamma := \partial\Omega$, $\Gamma_{\mathrm{S}} := \Gamma\backslash\Gamma_{\mathrm{D}}$, n is the outward-pointing unit normal vector to Γ and

$$\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) := \left\{v \in \mathrm{H}^1(\Omega,\mathbb{R}^d) \mid v = 0 \textit{ a.e. } \text{on } \Gamma_{\mathrm{D}}\right\}.$$

Recall that, in linear elasticity, A is the stiffness tensor, e is the infinitesimal strain tensor, σ_{n} is the normal stress, σ_τ is the shear stress, and f models volume forces (see Subsection 2.4 for details). The normal boundary condition on Γ_{S} is known as the Signorini unilateral conditions which described the non-permeability of Γ_{S} (that is $u_{\mathrm{n}} \leq 0$), and that there are only compressive stresses exerted on Γ_{S} (that is $\sigma_{\mathrm{n}}(u) \leq 0$). Note that we focus here on minimizing the energy functional (as in [15, 19, 40]) which corresponds to maximize the compliance (see [5]).

For any $\Omega \in \mathcal{U}_{\mathrm{ref}}$, the unique solution u_Ω to (SP_Ω) satisfies

$$\int_\Omega \mathrm{Ae}(u_\Omega) : \mathrm{e}(v - u_\Omega) \geq \int_\Omega f \cdot (v - u_\Omega), \qquad \forall v \in \mathcal{K}^1(\Omega),$$

where $\mathcal{K}^1(\Omega)$ is the nonempty closed convex subset of $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ given by

$$\mathcal{K}^1(\Omega) := \left\{v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) \mid v_{\mathrm{n}} \leq 0 \textit{ a.e. } \text{on } \Gamma_{\mathrm{S}}\right\},$$

and is characterized by $u_\Omega = \mathrm{proj}_{\mathcal{K}^1(\Omega)}(F_\Omega)$, where $F_\Omega \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the unique solution to the Dirichlet-Neumann problem

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(F)) = f & \text{in } \Omega, \\ F = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \mathrm{Ae}(F)\mathrm{n} = 0 & \text{on } \Gamma_{\mathrm{S}}, \end{cases}$$

and where $\mathrm{proj}_{\mathcal{K}^1(\Omega)}$ stands for the projection operator on $\mathcal{K}^1(\Omega)$. We refer for instance to [2] for details on existence/uniqueness and characterization of the solution to Problem (SP_Ω). In order to use our methodology, which is based, in particular, on the twice epi-differentiability and on the proximal operator, we characterize u_Ω as (see Remark 2.10)

$$u_\Omega = \mathrm{prox}_{\iota_{\mathcal{K}^1(\Omega)}}(F_\Omega),$$

where $\mathrm{prox}_{\iota_{\mathcal{K}^1(\Omega)}}$ is the proximal operator associated with the Signorini indicator function $\iota_{\mathcal{K}^1(\Omega)}$, which is defined by $\iota_{\mathcal{K}^1(\Omega)}(v) := 0$ if $v \in \mathcal{K}^1(\Omega)$, and $\iota_{\mathcal{K}^1(\Omega)}(v) := +\infty$ otherwise. To deal with the numerical treatment of the above shape optimization problem, a suitable expression of the shape gradient of $\mathcal{J}$ is required. To this aim, we follow the classical strategy developed in the shape optimization literature (see, e.g., [5, 20]). Consider $\Omega_0 \in \mathcal{U}_{\mathrm{ref}}$ and a direction $\theta \in \mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d,\mathbb{R}^d)$, where

$$\mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d,\mathbb{R}^d) := \left\{\theta \in \mathcal{C}^2(\mathbb{R}^d,\mathbb{R}^d) \cap \mathrm{W}^{2,\infty}(\mathbb{R}^d,\mathbb{R}^d) \mid \theta = 0 \text{ on } \Gamma_{\mathrm{D}}\right\}.$$

Then, for any $t \geq 0$ sufficiently small such that $\mathrm{id} + t\theta$ is a $\mathcal{C}^2$-diffeomorphism of $\mathbb{R}^d$, we denote by $\Omega_t := (\mathrm{id} + t\theta)(\Omega_0) \in \mathcal{U}_{\mathrm{ref}}$ and by $u_t := u_{\Omega_t} \in \mathrm{H}^1_{\mathrm{D}}(\Omega_t,\mathbb{R}^d)$, where $\mathrm{id} : \mathbb{R}^d \to \mathbb{R}^d$ stands for the identity operator. To get an expression of the shape gradient of $\mathcal{J}$ at Ω_0 in the direction θ, the first step naturally consists in obtaining an expression of the derivative of the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_t,\mathbb{R}^d)$ at $t = 0$. To overcome the issue that u_t is defined on the moving domain Ω_t, the classical change of variables $\mathrm{id}+t\theta$ is considered, and we prove that $\overline{u}_t := u_t \circ (\mathrm{id}+t\theta) \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ is the unique solution to the variational inequality

$$\int_{\Omega_0} \mathrm{J}_t \mathrm{A}\left[\nabla\overline{u}_t\,(\mathrm{I} + t\nabla\theta)^{-1}\right] : \nabla(v - \overline{u}_t)\,(\mathrm{I} + t\nabla\theta)^{-1} \geq \int_{\Omega_0} f_t \mathrm{J}_t \cdot (v - \overline{u}_t),$$

for all $v \in \mathrm{K}^1_t(\Omega_0) := \left\{v \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d) \mid v \cdot (\mathrm{I} + t\nabla\theta^\top)^{-1}\mathrm{n} \leq 0 \textit{ a.e. } \text{on } \Gamma_{\mathrm{S}_0}\right\}$, where n refers now to the outward-pointing unit normal vector to Γ_0, $f_t := f \circ (\mathrm{id}+t\theta) \in \mathrm{H}^1(\mathbb{R}^d,\mathbb{R}^d)$, $\mathrm{J}_t := \det(\mathrm{I}+t\nabla\theta) \in$

$\mathrm{L}^\infty(\mathbb{R}^d,\mathbb{R})$ is the Jacobian, $\nabla\theta$ stands for the standard Jacobian matrix of θ and I is the identity matrix of $\mathbb{R}^{d\times d}$. Now, to obtain an expression of the derivative of the map $t\in\mathbb{R}_+\mapsto\overline{u}_t\in\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ at $t=0$, which will be denoted by $\overline{u}'_0\in\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ and called material directional derivative, we write that $\overline{u}_t=\mathrm{prox}_{\iota_{\mathrm{K}^1_t(\Omega_0)}}(F_t)$, where $F_t\in\mathrm{H}^1(\Omega_0,\mathbb{R}^d)$ is the unique solution to the parameterized variational equality

$$\int_{\Omega_0}\mathrm{J}_t\mathrm{A}\left[\nabla F_t\left(\mathrm{I}+t\nabla\theta\right)^{-1}\right]:\nabla v\left(\mathrm{I}+t\nabla\theta\right)^{-1}=\int_{\Omega_0}f_t\mathrm{J}_t\cdot v,\qquad\forall v\in\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d),$$

and $\mathrm{prox}_{\iota_{\mathrm{K}^1_t(\Omega_0)}}$ is the proximal operator associated with the indicator function $\iota_{\mathrm{K}^1_t(\Omega_0)}$ considered on the Hilbert space $\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ endowed with a perturbed scalar product (see details in Subsection 3.1).

To deal with the differentiability (in a generalized sense) of the proximal operator, we use the same methodology already described in our paper [3], where we invoke the notion of twice epi-differentiability for convex functions (see [31]), which leads to the *protodifferentiability* of the corresponding proximal operators.

Let us emphasize that, in this paper, we do not prove theoretically the existence of a solution to the shape optimization problem (1.1). The interested reader can find some related existence results (for very specific geometries in the two dimensional case) in [17].

Main theoretical results. We summarize here our main theoretical results (given in Theorems 3.5 and 3.10). However we present the material and shape directional derivatives, and the shape gradient of $\mathcal{J}$ under some additional regularity assumptions, precisely in the framework of Corollaries 3.6, 3.8 and 3.11, because their expressions are more elegant in that case.

(i) Under some appropriate assumptions described in Corollary 3.6, the map $t\in\mathbb{R}_+\mapsto\overline{u}_t\in\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ is differentiable at $t=0$, and the material directional derivative $\overline{u}'_0\in\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ is the unique weak solution to the Signorini problem

$$\begin{cases}\begin{aligned}-\mathrm{div}(\mathrm{Ae}(\overline{u}'_0))&=\ell(\theta)&&\text{in }\Omega_0,\\ \overline{u}'_0&=0&&\text{on }\Gamma_{\mathrm{D}},\\ \sigma_\tau(\overline{u}'_0)&=h^m(\theta)_\tau&&\text{on }\Gamma_{\mathrm{S}_0},\\ \sigma_{\mathrm{n}}(\overline{u}'_0)&=h^m(\theta)_{\mathrm{n}}&&\text{on }\Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{N}}},\\ \overline{u}'_{0\mathrm{n}}&=(\nabla\theta u_0)_{\mathrm{n}}&&\text{on }\Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{D}}},\\ \overline{u}'_{0\mathrm{n}}\leq(\nabla\theta u_0)_{\mathrm{n}},\sigma_{\mathrm{n}}(\overline{u}'_0)\leq h^m(\theta)_{\mathrm{n}}\text{ and }\left(\overline{u}'_{0\mathrm{n}}-(\nabla\theta u_0)_{\mathrm{n}}\right)\left(\sigma_{\mathrm{n}}(\overline{u}'_0)-h^m(\theta)_{\mathrm{n}}\right)&=0&&\text{on }\Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{S}}},\end{aligned}\end{cases}$$

where $h^m(\theta):=((\mathrm{Ae}(u_0))\nabla\theta^\top+\mathrm{A}(\nabla u_0\nabla\theta)-\sigma_{\mathrm{n}}(u_0)(\mathrm{div}(\theta)\mathrm{I}+\nabla\theta^\top))\mathrm{n}\in\mathrm{L}^2(\Gamma_0,\mathbb{R}^d)$, $\ell(\theta)=-\mathrm{div}(\mathrm{Ae}(\nabla u_0\theta))\in\mathrm{L}^2(\Omega_0,\mathbb{R}^d)$ and Γ_{S_0} is decomposed, up to a null set, as $\Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{N}}}\cup\Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{D}}}\cup\Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{S}}}$ (see details in Corollary 3.6).

(ii) We deduce in Corollary 3.8 that, under appropriate assumptions, the shape directional derivative, defined by $u'_0:=\overline{u}'_0-\nabla u_0\theta\in\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ (which corresponds, roughly speaking, to the derivative of the map $t\in\mathbb{R}_+\mapsto u_t\in\mathrm{H}^1_{\mathrm{D}}(\Omega_t,\mathbb{R}^d)$ at $t=0$), is the unique weak solution to the Signorini problem

$$\begin{cases}\begin{aligned}-\mathrm{div}(\mathrm{Ae}(u'_0))&=0&&\text{in }\Omega_0,\\ u'_0&=0&&\text{on }\Gamma_{\mathrm{D}},\\ \sigma_\tau(u'_0)&=h^s(\theta)_\tau&&\text{on }\Gamma_{\mathrm{S}_0},\\ \sigma_{\mathrm{n}}(u'_0)&=h^s(\theta)_{\mathrm{n}}&&\text{on }\Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{N}}},\\ u'_{0\mathrm{n}}&=W(\theta)_{\mathrm{n}}&&\text{on }\Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{D}}},\\ u'_{0\mathrm{n}}\leq W(\theta)_{\mathrm{n}},\sigma_{\mathrm{n}}(u'_0)\leq h^s(\theta)_{\mathrm{n}}\text{ and }\left(u'_{0\mathrm{n}}-W(\theta)_{\mathrm{n}}\right)\left(\sigma_{\mathrm{n}}(u'_0)-h^s(\theta)_{\mathrm{n}}\right)&=0&&\text{on }\Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{S}}},\end{aligned}\end{cases}$$

where $W(\theta) := (\nabla\theta u_0) - (\nabla u_0\theta) \in \mathrm{H}^{1/2}(\Gamma_0, \mathbb{R}^d)$,

$$h^s(\theta) := \theta \cdot \mathrm{n}\left(\partial_{\mathrm{n}}\left(\mathrm{Ae}(u_0)\mathrm{n}\right) - \partial_{\mathrm{n}}\left(\mathrm{Ae}(u_0)\right)\mathrm{n}\right) + \mathrm{Ae}(u_0)\nabla_\tau\left(\theta\cdot\mathrm{n}\right) \\ - \nabla(\mathrm{Ae}(u_0)\mathrm{n})\theta - \sigma_{\mathrm{n}}(u_0)\left(\mathrm{div}_\tau(\theta)\mathrm{I} + \nabla\theta^\top\right)\mathrm{n} \in \mathrm{L}^2(\Gamma_0, \mathbb{R}^d),$$

and where $\partial_{\mathrm{n}}\left(\mathrm{Ae}(u_0)\mathrm{n}\right) := \nabla(\mathrm{Ae}(u_0)\mathrm{n})\mathrm{n}$ is the normal derivative of $\mathrm{Ae}(u_0)\mathrm{n}$, and $\partial_{\mathrm{n}}\left(\mathrm{Ae}(u_0)\right)$ is the matrix whose the i-th line is the vector $\partial_{\mathrm{n}}\left(\mathrm{Ae}(u_0)_i\right) := \nabla(\mathrm{Ae}(u_0)_i)\mathrm{n}$, where $\mathrm{Ae}(u_0)_i$ is the i-th line of the matrix $\mathrm{Ae}(u_0)$, for all $i \in [[1, d]]$.

(iii) Finally the two previous items are used to obtain Corollary 3.11 asserting that, under appropriate assumptions, the shape gradient of $\mathcal{J}$ at Ω_0 in the direction θ is given by

$$\mathcal{J}'(\Omega_0)(\theta) = \int_{\Gamma_{\mathrm{S}_0}} \left(\theta\cdot\mathrm{n}\left(\frac{\mathrm{Ae}(u_0):\mathrm{e}(u_0)}{2} - f\cdot u_0\right) + \mathrm{Ae}(u_0)\mathrm{n}\cdot(\nabla\theta u_0 - \nabla u_0\theta)\right).$$

One can notice that $\mathcal{J}'(\Omega_0)$ depends only on u_0 (and not on u_0'), thus its expression is explicit and linear with respect to the direction θ and allows us to exhibit a descent direction of $\mathcal{J}$ (see Section 4 for details). Hence, using this descent direction together with a basic Uzawa algorithm to take into account the volume constraint, we perform in Section 4 numerical simulations to solve the shape optimization problem (1.1) on a two-dimensional example.

Organization of the paper. The paper is organized as follows. Section 2 is dedicated to some basic recalls from convex, variational and functional analysis, capacity theory, differential geometry and boundary value problems involved all along the paper. Section 3 is the core of the present work where the main theoretical results are stated and proved. Finally, in Section 4, numerical simulations are performed to solve the shape optimization problem (1.1) on a two-dimensional example.

2 Reminders

In this section we start with some notions from convex, variational and functional analysis in Subsection 2.1, some concepts from capacity theory in Subsection 2.2, some results on differential geometry in Subsection 2.3, and then we conclude with some reminders on boundary value problems in Subsection 2.4.

2.1 Notions from convex, variational and functional analysis

For notions and results presented in this section, we refer to standard references such as [8, 27, 30] and [33, Chapter 12]. In what follows $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ stands for a general real Hilbert space.

Definition 2.1 (Polar cone)**.** *Let* P *be a nonempty subset of* $\mathcal{H}$. *The polar cone to* P *is the nonempty closed convex subset of* $\mathcal{H}$ *defined by*

$$\mathrm{P}^\circ := \left\{z \in \mathcal{H} \mid \langle z, p\rangle_{\mathcal{H}} \leq 0, \forall p \in \mathrm{P}\right\}.$$

Definition 2.2 (Normal cone)**.** *Let* C *be a nonempty closed convex subset of* $\mathcal{H}$ *and* $x \in \mathrm{C}$. *The normal cone to* C *at* x *is the nonempty closed convex cone of* $\mathcal{H}$ *defined by*

$$\mathrm{N}_{\mathrm{C}}(x) := \left\{z \in \mathcal{H} \mid \langle z, c - x\rangle_{\mathcal{H}} \leq 0, \forall c \in \mathrm{C}\right\}.$$

Definition 2.3 (Tangent cone)**.** *Let* C *be a nonempty closed convex subset of* $\mathcal{H}$ *and* $x \in \mathrm{C}$*. The tangent cone to* C *at* x *is the nonempty closed convex cone of* $\mathcal{H}$ *defined by*

$$\mathrm{T}_{\mathrm{C}}(x) := \overline{\{z \in \mathcal{H} \mid \exists \lambda > 0, x + \lambda z \in \mathrm{C}\}}.$$

Definition 2.4 (Polyhedric set)**.** *Let* C *be a nonempty closed convex subset of* $\mathcal{H}$*. We say that* C *is polyhedric at* $x \in \mathrm{C}$ *for* $y \in \mathrm{N}_{\mathrm{C}}(x)$ *if*

$$\mathrm{T}_{\mathrm{C}}(x) \cap (\mathbb{R}y)^{\perp} = \overline{\{z \in \mathcal{H} \mid \exists \lambda > 0, x + \lambda z \in \mathrm{C}\} \cap (\mathbb{R}y)^{\perp}}.$$

Remark 2.5. Recall that, in finite dimension, polyhedric sets reduce to polyhedral sets, which is the intersection of a finite set of closed half-spaces (see, e.g., [24]).

Definition 2.6 (Domain and epigraph)**.** *Let* $\phi : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$*. The domain and the epigraph of* ϕ *are respectively defined by*

$$\operatorname{dom}(\phi) := \{x \in \mathcal{H} \mid \phi(x) < +\infty\} \quad \text{and} \quad \operatorname{epi}(\phi) := \{(x,t) \in \mathcal{H} \times \mathbb{R} \mid \phi(x) \leq t\}.$$

Recall that $\phi : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *proper* if $\operatorname{dom}(\phi) \neq \emptyset$ and $\phi(x) > -\infty$ for all $x \in \mathcal{H}$. Moreover, ϕ is a convex (resp. lower semi-continuous) function on $\mathcal{H}$ if and only if $\operatorname{epi}(\phi)$ is a convex (resp. closed) subset of $\mathcal{H} \times \mathbb{R}$.

Definition 2.7 (Convex subdifferential operator)**.** *Let* $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ *be a proper function. We denote by* $\partial\phi : \mathcal{H} \rightrightarrows \mathcal{H}$ *the convex subdifferential operator of* ϕ*, defined by*

$$\partial\phi(x) := \{y \in \mathcal{H} \mid \forall z \in \mathcal{H},\ \langle y, z - x\rangle_{\mathcal{H}} \leq \phi(z) - \phi(x)\},$$

for all $x \in \mathcal{H}$*.*

Example 2.8. *Let* C *be a nonempty closed convex subset of* $\mathcal{H}$*, and* ι_{C} *be the indicator function of* C*, defined by* $\iota_{\mathrm{C}}(x) := 0$ *if* $x \in \mathrm{C}$*, and* $\iota_{\mathrm{C}}(x) := +\infty$ *otherwise. Then, for all* $x \in \mathrm{C}$*,*

$$\partial\iota_{\mathrm{C}}(x) = \mathrm{N}_{\mathrm{C}}(x).$$

Definition 2.9 (Proximal operator)**.** *Let* $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semi-continuous convex function. The proximal operator associated with* ϕ *is the map* $\operatorname{prox}_{\phi} : \mathcal{H} \to \mathcal{H}$ *defined by*

$$\operatorname{prox}_{\phi}(x) := \underset{y \in \mathcal{H}}{\operatorname{argmin}} \left[\phi(y) + \frac{1}{2}\|y - x\|_{\mathcal{H}}^{2}\right] = (\mathrm{I} + \partial\phi)^{-1}(x),$$

for all $x \in \mathcal{H}$*, where* $\mathrm{I} : \mathcal{H} \to \mathcal{H}$ *stands for the identity operator.*

Remark 2.10. *Note that, if* $\phi := \iota_{\mathrm{C}}$*, where* ι_{C} *is the indicator function of a nonempty closed convex subset* $\mathrm{C} \subset \mathcal{H}$*, then* ι_{C} *is a proper lower semi-continuous convex function and*

$$\operatorname{prox}_{\iota_{\mathrm{C}}} = \operatorname{proj}_{\mathrm{C}},$$

where $\operatorname{proj}_{\mathrm{C}}$ *is the projection operator on* C*.*

It is well-known that, if $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semi-continuous convex function, then $\partial\phi$ is a maximal monotone operator (see, e.g., [30]), and thus the proximal operator $\operatorname{prox}_{\phi}$ is well-defined, single-valued and nonexpansive, i.e. Lipschitz continuous with modulus 1 (see, e.g., [8, Chapter II]).

As mentioned in Introduction, the unique solution to the Signorini problem considered in this paper can be expressed via the proximal operator of the associated Signorini indicator function $\iota_{\mathcal{K}^1(\Omega)}$. Therefore the shape sensitivity analysis of this problem is related to the differentiability (in a generalized sense) of the involved proximal operator. To investigate this issue, we will use the notion of twice epi-differentiability (see [31]) defined as the Mosco epi-convergence of second-order difference quotient functions. Our aim in what follows is to provide reminders and backgrounds on these notions for the reader's convenience. For more details, we refer to [33, Chapter 7, Section B p.240] for the finite-dimensional case and to [13] for the infinite-dimensional case. The strong (resp. weak) convergence of a sequence in $\mathcal{H}$ will be denoted by $\rightarrow$ (resp. $\rightharpoonup$) and note that all limits with respect to t will be considered for $t \rightarrow 0^+$.

Definition 2.11 (Mosco-convergence)**.** *The outer, weak-outer, inner and weak-inner limits of a parameterized family $(A_t)_{t>0}$ of subsets of $\mathcal{H}$ are respectively defined by*

$$\begin{aligned}
\limsup A_t &:= \left\{x \in \mathcal{H} \mid \exists (t_n)_{n\in\mathbb{N}} \rightarrow 0^+, \exists \left(x_n\right)_{n\in\mathbb{N}} \rightarrow x, \forall n \in \mathbb{N}, x_n \in A_{t_n}\right\},\\
\text{w-}\limsup A_t &:= \left\{x \in \mathcal{H} \mid \exists (t_n)_{n\in\mathbb{N}} \rightarrow 0^+, \exists \left(x_n\right)_{n\in\mathbb{N}} \rightharpoonup x, \forall n \in \mathbb{N}, x_n \in A_{t_n}\right\},\\
\liminf A_t &:= \left\{x \in \mathcal{H} \mid \forall (t_n)_{n\in\mathbb{N}} \rightarrow 0^+, \exists \left(x_n\right)_{n\in\mathbb{N}} \rightarrow x, \exists N \in \mathbb{N}, \forall n \geq N, x_n \in A_{t_n}\right\},\\
\text{w-}\liminf A_t &:= \left\{x \in \mathcal{H} \mid \forall (t_n)_{n\in\mathbb{N}} \rightarrow 0^+, \exists \left(x_n\right)_{n\in\mathbb{N}} \rightharpoonup x, \exists N \in \mathbb{N}, \forall n \geq N, x_n \in A_{t_n}\right\}.
\end{aligned}$$

The family $(A_t)_{t>0}$ is said to be Mosco-convergent if $\text{w-}\limsup A_t \subset \liminf A_t$*. In that case, all the previous limits are equal and we write*

$$\text{M-lim}A_t := \liminf A_t = \limsup A_t = \text{w-}\liminf A_t = \text{w-}\limsup A_t.$$

Definition 2.12 (Mosco epi-convergence)**.** *Let $(\phi_t)_{t>0}$ be a parameterized family of functions $\phi_t : \mathcal{H} \rightarrow \mathbb{R}\cup\{\pm\infty\}$ for all $t > 0$. We say that $(\phi_t)_{t>0}$ is Mosco epi-convergent if $(\mathrm{epi}(\phi_t))_{t>0}$ is Mosco-convergent in $\mathcal{H} \times \mathbb{R}$. Then we denote by $\text{ME-lim } \phi_t : \mathcal{H} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ the function characterized by its epigraph $\mathrm{epi}\,(\text{ME-lim } \phi_t) := \text{M-lim epi}\,(\phi_t)$ and we say that $(\phi_t)_{t>0}$ Mosco epi-converges to $\text{ME-lim } \phi_t$.*

Now let us recall the notion of twice epi-differentiability introduced by R.T. Rockafellar in 1985 (see [31]) that generalizes the classical notion of second-order derivative to nonsmooth convex functions.

Definition 2.13 (Twice epi-differentiability)**.** *A proper lower semi-continuous convex function $\phi : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be twice epi-differentiable at $x \in \mathrm{dom}(\phi)$ for $y \in \partial\phi(x)$ if the family of second-order difference quotient functions $(\delta_t^2\phi(x|y))_{t>0}$ defined by*

$$\begin{array}{rccl}
\delta_t^2\phi(x|y): & \mathcal{H} & \longrightarrow & \mathbb{R}\cup\{+\infty\}\\
& z & \longmapsto & \dfrac{\phi(x+tz)-\phi(x)-t\,\langle y,z\rangle_{\mathcal{H}}}{\frac{1}{2}t^2},
\end{array}$$

for all $t > 0$, is Mosco epi-convergent. In that case we denote by

$$\mathrm{d}_e^2\phi(x|y) := \text{ME-lim } \delta_t^2\phi(x|y),$$

which is called the second-order epi-derivative of ϕ at x for y.

The following result is extracted from [13, Chapter 2, Example 2.10 p.287].

Lemma 2.14. *Let C be a nonempty closed convex subset of $\mathcal{H}$. If C is polyhedric at $x \in \mathrm{C}$ for $y \in \mathrm{N}_\mathrm{C}(x)$, then ι_C is twice epi-differentiable at x for y and*

$$\mathrm{d}_e^2\iota_\mathrm{C}(x|y) = \iota_{\mathrm{T}_\mathrm{C}(x)\cap(\mathbb{R}y)^\perp},$$

where $\mathrm{N}_\mathrm{C}(x)$ (resp. $\mathrm{T}_\mathrm{C}(x)$) is the normal cone (resp. tangent cone) to C at x.

We conclude this section with a last proposition (see, e.g., [32, 33] for the finite-dimensional case and [1, 13] for the infinite-dimensional one). We bring to the attention of the reader that Proposition 2.15 is the key point in order to derive our main results.

Proposition 2.15. *Let $\Phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function on $\mathcal{H}$. Let $F : \mathbb{R}^+ \to \mathcal{H}$ and let $u : \mathbb{R}^+ \to \mathcal{H}$ be defined by*

$$u(t) := \mathrm{prox}_{\Phi}(F(t)),$$

for all $t \geq 0$. If the conditions

1. *F is differentiable at $t = 0$;*

2. *Φ is twice epi-differentiable at $u(0)$ for $F(0) - u(0) \in \partial\Phi(u(0))$;*

are both satisfied, then u is differentiable at $t = 0$ with

$$u'(0) = \mathrm{prox}_{\mathrm{d}_e^2\Phi(u(0)|F(0)-u(0))}(F'(0)).$$

2.2 Notions from capacity theory

Let us recall some notions from capacity theory (we refer to standard references such as [12, 16, 20, 26]). Let us consider $(\mathrm{X}, \mathcal{B}(\mathrm{X}), \xi)$ be a positively measured topological space with its borelian σ-algebra, ξ a Radon measure, and where $\mathrm{X} \subset \mathbb{R}^d$ is a locally compact set, admitting a countable compact covering. Let $\mathcal{H} \subset \mathrm{L}^2(\mathrm{X}, \xi)$ be a vector space endowed with a scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ the corresponding norm.

Definition 2.16. *Consider $\mathrm{B} \in \mathcal{B}(\mathrm{X})$ and let us introduce the closed convex subset*

$$\mathrm{C}_{\mathrm{B}} := \{v \in \mathcal{H} \mid v \geq 1 \ \xi\text{-a.e. on a neighborhood of } \mathrm{B}\}.$$

The capacity of B is defined by

$$\mathrm{cap}(\mathrm{B}) := \left\|\mathrm{proj}_{\mathrm{C}_{\mathrm{B}}}(0)\right\|_{\mathcal{H}}^2,$$

where $\mathrm{proj}_{\mathrm{C}_{\mathrm{B}}}$ is the projection operator onto the nonempty closed convex set C_{B}.

Definition 2.17. *A property holds quasi everywhere (denoted q.e.) if it holds for all elements in a set except a subset of null capacity.*

Definition 2.18. *A function $v : \mathrm{X} \to \mathbb{R}$ is said to be quasi-continuous if there exists a decreasing sequence of open sets $(w_n)_{n\in\mathbb{N}}$ such that $\mathrm{cap}(w_n) \to 0$ when $n \to +\infty$ and $v_{|\mathrm{X}\backslash w_n}$ is continuous for all $n \in \mathbb{N}$.*

Now, let us assume that $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is a Dirichlet space (see, e.g., [26]). Then, one can prove the following propositions (see, e.g., [16, 20, 26]).

Proposition 2.19. *For all $v \in \mathcal{H}$, there exists a unique quasi-continuous representative in the class of v (for the q.e. equivalence relation).*

To conclude, let us give two examples of Dirichlet space (see [26] for the first example and [39, Chapter 4] for the second one).

Example 2.20. *Let Ω is a nonempty bounded connected open subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary. Then $\mathcal{H} := \mathrm{H}^1(\Omega, \mathbb{R})$ endowed with its standard scalar product $\langle \cdot, \cdot \rangle_{\mathrm{H}^1(\Omega,\mathbb{R})}$ is a Dirichlet space.*

Example 2.21. *Let Ω be a nonempty bounded connected open subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\Gamma := \partial\Omega$. Assume that Γ is given by the decomposition $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are two measurable disjoint subsets of Γ. Then,*

$$\mathcal{H} := \left\{ v \cdot \mathrm{n} \in \mathrm{H}^{1/2}(\Gamma_2, \mathbb{R}) \mid v \in \mathrm{H}^1(\Omega, \mathbb{R}^d) \text{ and } v = 0 \text{ a.e. on } \Gamma_1 \right\},$$

is a Dirichlet space endowed with the scalar product defined in [39, Chapter 4, Eq. (4.192) p.208], where n *is the outward-pointing unit normal vector to* Γ.

2.3 Reminders on differential geometry

Let $d \in \mathbb{N}^*$ be a positive integer, Ω be a nonempty bounded connected open subset of $\mathbb{R}^d$ with a Lipschitz-boundary $\Gamma := \partial\Omega$ and n the outward-pointing unit normal vector to Γ. In the whole paper we denote by $\mathcal{D}(\Omega, \mathbb{R}^d)$ the set of functions that are infinitely differentiable with compact support in Ω, by $\mathcal{D}'(\Omega, \mathbb{R}^d)$ the set of distributions on Ω, for $(m, p) \in \mathbb{N} \times \mathbb{N}^*$, by $\mathrm{W}^{m,p}(\Omega, \mathbb{R}^d)$, $\mathrm{L}^2(\Gamma, \mathbb{R}^d)$, $\mathrm{H}^{1/2}(\Gamma, \mathbb{R}^d)$, $\mathrm{H}^{-1/2}(\Gamma, \mathbb{R}^d)$, the usual Lebesgue and Sobolev spaces endowed with their standard norms, and we denote by $\mathrm{H}^m(\Omega, \mathbb{R}^d) := \mathrm{W}^{m,2}(\Omega, \mathbb{R}^d)$ and by $\mathrm{H}_{\mathrm{div}}(\Omega, \mathbb{R}^{d\times d}) := \{w \in \mathrm{L}^2(\Omega, \mathbb{R}^{d\times d}) \mid \mathrm{div}(w) \in \mathrm{L}^2(\Omega, \mathbb{R}^d)\}$.

The next proposition, known as divergence formula, can be found in [4, Theorem 4.4.7 p.104].

Proposition 2.22 (Divergence formula)**.** *If $w \in \mathrm{H}_{\mathrm{div}}(\Omega, \mathbb{R}^{d\times d})$, then w admits a normal trace, denoted by $w\mathrm{n} \in \mathrm{H}^{-1/2}(\Gamma, \mathbb{R}^d)$, satisfying*

$$\int_\Omega \mathrm{div}(w) \cdot v + \int_\Omega w : \nabla v = \langle w\mathrm{n}, v \rangle_{\mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)\times \mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)}, \qquad \forall v \in \mathrm{H}^1(\Omega, \mathbb{R}^d).$$

The following propositions will be useful and their proofs can be found in [20].

Proposition 2.23. *Assume that Γ is of class $\mathcal{C}^2$ and let $\theta \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$. It holds that*

$$\int_\Gamma (\theta \cdot \nabla v + v\mathrm{div}_\tau(\theta)) = \int_\Gamma \theta \cdot \mathrm{n}(\partial_\mathrm{n} v + Hv), \qquad \forall v \in \mathrm{W}^{2,1}(\Omega, \mathbb{R}),$$

where $\mathrm{div}_\tau(\theta) := \mathrm{div}(\theta) - (\nabla\theta\mathrm{n} \cdot \mathrm{n}) \in \mathrm{L}^\infty(\Gamma)$ is the tangential divergence of θ, $\partial_\mathrm{n} v := \nabla v \cdot \mathrm{n} \in \mathrm{L}^1(\Gamma, \mathbb{R})$ stands for the normal derivative of v, and H stands for the mean curvature of Γ.

Proposition 2.24. *Assume that Γ is of class $\mathcal{C}^2$ and let $w \in \mathrm{H}^2(\Omega, \mathbb{R}^{d\times d})$. It holds that*

$$\mathrm{div}(w) = \mathrm{div}_\tau (w_\tau) + Hw\mathrm{n} + (\partial_\mathrm{n} w)\,\mathrm{n} \qquad \text{a.e. on } \Gamma,$$

where $\mathrm{div}_\tau (w_\tau) \in \mathrm{L}^2(\Gamma, \mathbb{R}^d)$ is the vector whose the i-th component is defined by $(\mathrm{div}_\tau (w_\tau))_i := \mathrm{div}_\tau((w_i)_\tau) \in \mathrm{L}^2(\Gamma, \mathbb{R})$, where $(w_i)_\tau := w_i - (w_i \cdot \mathrm{n})\mathrm{n} \in \mathrm{L}^2(\Gamma, \mathbb{R}^d)$ and $w_i \in \mathbb{R}^d$ is the i-th line of w, and where $\partial_\mathrm{n} w \in \mathrm{L}^2(\Gamma, \mathbb{R}^{d\times d})$ is the matrix whose the i-th line is the vector $\partial_\mathrm{n} w_i := (\nabla w_i)\mathrm{n} \in \mathrm{L}^2(\Gamma, \mathbb{R}^d)$, for all $i \in [[1, d]]$. Moreover it holds that

$$\int_\Gamma v \cdot \mathrm{div}_\tau (w_\tau) = -\int_\Gamma w : \nabla_\tau v, \qquad \forall v \in \mathrm{H}^2(\Omega, \mathbb{R}^d),$$

where $\nabla_\tau v$ is the matrix whose the i-th line is the tangential gradient $\nabla_\tau v_i := \nabla v_i - (\partial_\mathrm{n} v_i)\mathrm{n} \in \mathrm{H}^{1/2}(\Gamma, \mathbb{R}^d)$, for all $i \in [[1, d]]$.

2.4 Some required boundary value problems

As mentioned in Introduction, the major part of the present work consists in performing the sensitivity analysis of a Signorini problem with respect to shape perturbation. To this aim one has to recall some classical boundary value problems: a Dirichlet-Neumann problem and a Signorini problem. Our aim in this subsection is to recall basic notions and results concerning these boundary value problems for the reader's convenience. Since the proofs are very similar to the ones detailed in the paper [2], they will be omitted here.

Let $d \in \{2,3\}$ and Ω be a nonempty bounded connected open subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\Gamma := \partial\Omega$. Let us assume that Γ is decomposed as $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}}$, where Γ_{D} and Γ_{S} are two measurable pairwise disjoint subsets of Γ, such that Γ_{D} has a positive measure. In that case,

$$\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) := \left\{ v \in \mathrm{H}^1(\Omega,\mathbb{R}^d) \mid v = 0 \ \textit{a.e.} \text{ on } \Gamma_{\mathrm{D}} \right\},$$

is a linear subspace of $\mathrm{H}^1(\Omega,\mathbb{R}^d)$.

Moreover, we assume that Ω is an elastic solid satisfying the linear elastic model (see, e.g., [36])

$$\sigma(v) = \mathrm{Ae}(v),$$

where σ is the Cauchy stress tensor, A the stiffness tensor, and e is the infinitesimal strain tensor defined by

$$\mathrm{e}(v) := \frac{1}{2}(\nabla v + \nabla v^\top),$$

for all displacement field $v \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$. We also assume that all coefficients of A are constant (denoted by a_{ijkl} for all $(i,j,k,l) \in \{1,...,d\}^4$), and there exists one constant $\alpha > 0$ such that all coefficients of A and e (denoted by ϵ_{ij} for all $(i,j) \in \{1,...,d\}^2$) satisfy

$$a_{ijkl} = a_{jikl} = a_{lkij}, \ \sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{d}\sum_{l=1}^{d} a_{ijkl}\epsilon_{ij}(v_1)(x)\epsilon_{kl}(v_2)(x) \geq \alpha \sum_{i=1}^{d}\sum_{j=1}^{d} \epsilon_{ij}(v_1)(x)\epsilon_{ij}(v_2)(x),$$

for all displacement field $v_1, v_2 \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$ and for almost all $x \in \Omega$. Thus,

$$\begin{array}{rccc} \langle\cdot,\cdot\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} : & \left(\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\right)^2 & \longrightarrow & \mathbb{R} \\ & (v_1,v_2) & \longmapsto & \displaystyle\int_\Omega \mathrm{Ae}(v_1) : \mathrm{e}(v_2), \end{array}$$

is a scalar product on $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ (see, e.g., [14, Chapter 3]), and we denote by $\|\cdot\|_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)}$ the corresponding norm. Moreover, from the assumptions on A, note that $\mathrm{Ae}(v) = \mathrm{A}\nabla v$, for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$.

We denote by n the outward-pointing unit normal vector to Γ. Therefore, for any $v \in \mathrm{L}^2(\Gamma,\mathbb{R}^d)$, one has $v = v_{\mathrm{n}}\mathrm{n} + v_\tau$, where $v_{\mathrm{n}} := v \cdot \mathrm{n} \in \mathrm{L}^2(\Gamma,\mathbb{R})$ and $v_\tau := v - v_{\mathrm{n}}\mathrm{n} \in \mathrm{L}^2(\Gamma,\mathbb{R}^d)$. In particular, if for some $v \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$, the stress vector $\mathrm{Ae}(v)\mathrm{n}$ is in $\mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R}^d)$, then we use the notation

$$\mathrm{Ae}(v)\mathrm{n} = \sigma_{\mathrm{n}}(v)\mathrm{n} + \sigma_\tau(v),$$

where $\sigma_{\mathrm{n}}(v) \in \mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R})$ is the normal stress and $\sigma_\tau \in \mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R}^d)$ the shear stress.

We also denote, for all $(x,y,\mathrm{M}) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$, by $xy^\top$ the matrix whose the i-th line is given by the vector $x_i y$, where $x_i \in \mathbb{R}$ is the i-th component of x, and by $\mathrm{div}(\mathrm{M})$ the vector whose the i-th component is defined by $(\mathrm{div}(\mathrm{M}))_i := \mathrm{div}(\mathrm{M}_i)$, where M_i is the i-th line of M, for all $i \in [[1,d]]$.

In the sequel, consider $k \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$, $h \in \mathrm{L}^2(\Gamma_{\mathrm{S}},\mathbb{R}^d)$ and $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$.

2.4.1 A Problem with Dirichlet-Neumann Conditions

Consider the Dirichlet-Neumann problem given by

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(F)) = k & \text{in } \Omega, \\ F = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \mathrm{Ae}(F)\mathrm{n} = h & \text{on } \Gamma_{\mathrm{S}}, \end{cases} \tag{DN}$$

where the data are given at the beginning of Subsection 2.4.

Definition 2.25 (Strong solution to the Dirichlet-Neumann problem)**.** *A (strong) solution to the Dirichlet-Neumann problem* (DN) *is a function* $F \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ *such that* $-\mathrm{div}(\mathrm{Ae}(F)) = k$ *in* $\mathcal{D}'(\Omega, \mathbb{R}^d)$*,* $F = 0$ *a.e. on* Γ_{D}*,* $\mathrm{Ae}(F)\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$ *with* $\mathrm{Ae}(F)\mathrm{n} = h$ *a.e. on* Γ_{S}*.*

Definition 2.26 (Weak solution to the Dirichlet-Neumann problem)**.** *A weak solution to the Dirichlet-Neumann problem* (DN) *is a function* $F \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ *such that*

$$\int_\Omega \mathrm{Ae}(F) : \mathrm{e}(v) = \int_\Omega k \cdot v + \int_{\Gamma_{\mathrm{S}}} h \cdot v, \qquad \forall v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d).$$

Proposition 2.27. *A function* $F \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ *is a (strong) solution to the Dirichlet-Neumann problem* (DN) *if and only if* F *is a weak solution to the Dirichlet-Neumann problem* (DN)*.*

Using the Riesz representation theorem, we obtain the following existence/uniqueness result.

Proposition 2.28. *The Dirichlet-Neumann problem* (DN) *admits a unique solution* $F \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$*.*

2.4.2 A Signorini problem

In this part, let us assume that Γ_{S} is decomposed, up to a null set, as $\Gamma_{\mathrm{S_N}} \cup \Gamma_{\mathrm{S_D}} \cup \Gamma_{\mathrm{S_S}}$, where $\Gamma_{\mathrm{S_N}}$, $\Gamma_{\mathrm{S_D}}$ and $\Gamma_{\mathrm{S_S}}$ are three measurable pairwise disjoint subsets of Γ_{S}. Consider the Signorini problem given by

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(u)) = k & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \sigma_\tau(u) = h_\tau & \text{on } \Gamma_{\mathrm{S}}, \\ \sigma_{\mathrm{n}}(u) = h_{\mathrm{n}} & \text{on } \Gamma_{\mathrm{S_N}}, \\ u_{\mathrm{n}} = w_{\mathrm{n}} & \text{on } \Gamma_{\mathrm{S_D}}, \\ u_{\mathrm{n}} \leq w_{\mathrm{n}},\ \sigma_{\mathrm{n}}(u) \leq h_{\mathrm{n}} \text{ and } (u_{\mathrm{n}} - w_{\mathrm{n}})(\sigma_{\mathrm{n}}(u) - h_{\mathrm{n}}) = 0 & \text{on } \Gamma_{\mathrm{S_S}}, \end{cases} \tag{MSP}$$

where the data are given at the beginning of Subsection 2.4.

Definition 2.29 (Strong solution)**.** *A (strong) solution to the problem* (MSP) *is a function* $u \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ *such that* $-\mathrm{div}(\mathrm{Ae}(u)) = k$ *in* $\mathcal{D}'(\Omega, \mathbb{R}^d)$*,* $u = 0$ *a.e. on* Γ_{D}*,* $u_{\mathrm{n}} = w_{\mathrm{n}}$ *a.e. on* $\Gamma_{\mathrm{S_D}}$*,* $\mathrm{Ae}(u)\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$ *with* $\sigma_\tau(u) = h_\tau$ *a.e. on* Γ_{S}*,* $\sigma_{\mathrm{n}} = h_{\mathrm{n}}$ *a.e. on* $\Gamma_{\mathrm{S_N}}$*,* $u_{\mathrm{n}} \leq w_{\mathrm{n}}$*,* $\sigma_{\mathrm{n}}(u) \leq h_{\mathrm{n}}$ *and* $(u_{\mathrm{n}} - w_{\mathrm{n}})(\sigma_{\mathrm{n}}(u) - h_{\mathrm{n}}) = 0$ *a.e. on* $\Gamma_{\mathrm{S_S}}$*.*

Definition 2.30 (Weak solution)**.** *A weak solution to problem* (MSP) *is a function* $u \in \mathcal{K}^1_w(\Omega)$ *such that*

$$\int_\Omega \mathrm{Ae}(u) : \mathrm{e}(v - u) \geq \int_\Omega k \cdot (v - u) + \int_{\Gamma_{\mathrm{S}}} h \cdot (v - u), \qquad \forall v \in \mathcal{K}^1_w(\Omega),$$

where $\mathcal{K}^1_w(\Omega)$ *is the nonempty closed convex subset of* $\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ *defined by*

$$\mathcal{K}^1_w(\Omega) := \left\{ v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) \mid v_{\mathrm{n}} = w_{\mathrm{n}} \text{ a.e. on } \Gamma_{\mathrm{S_D}} \text{ and } v_{\mathrm{n}} \leq w_{\mathrm{n}} \text{ a.e. on } \Gamma_{\mathrm{S_S}} \right\}.$$

One can prove that a (strong) solution is a weak solution but, to the best of our knowledge, without additional assumption, one cannot prove the converse. To get the equivalence, we need to assume, in particular, that the decomposition $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S_N}} \cup \Gamma_{\mathrm{S_D}} \cup \Gamma_{\mathrm{S_S}}$ of Γ is *consistent* in the following sense.

Definition 2.31 (Consistent decomposition)**.** *The decomposition $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S_N}} \cup \Gamma_{\mathrm{S_D}} \cup \Gamma_{\mathrm{S_S}}$ of Γ is said to be consistent if*

1. *for almost all $s \in \Gamma_{\mathrm{S_S}}$, $s \in \mathrm{int}_{\Gamma}(\Gamma_{\mathrm{S_S}})$;*

2. *the nonempty closed convex subset $\mathcal{K}_w^{1/2}(\Gamma)$ of $\mathrm{H}^{1/2}(\Gamma, \mathbb{R}^d)$ defined by*

$$\mathcal{K}_w^{1/2}(\Gamma) := \Big\{ v \in \mathrm{H}^{1/2}(\Gamma, \mathbb{R}^d) \mid v = 0 \text{ a.e. on } \Gamma_{\mathrm{D}},\ v_{\mathrm{n}} = w_{\mathrm{n}} \text{ a.e. on } \Gamma_{\mathrm{S_D}} \\ \text{and } v_{\mathrm{n}} \leq w_{\mathrm{n}} \text{ a.e. on } \Gamma_{\mathrm{S_S}} \Big\},$$

is dense in the nonempty closed convex subset $\mathcal{K}_w^{0}(\Gamma)$ of $\mathrm{L}^2(\Gamma, \mathbb{R}^d)$ defined by

$$\mathcal{K}_w^{0}(\Gamma) := \Big\{ v \in \mathrm{L}^{2}(\Gamma, \mathbb{R}^d) \mid v = 0 \text{ a.e. on } \Gamma_{\mathrm{D}},\ v_{\mathrm{n}} = w_{\mathrm{n}} \text{ a.e. on } \Gamma_{\mathrm{S_D}} \\ \text{and } v_{\mathrm{n}} \leq w_{\mathrm{n}} \text{ a.e. on } \Gamma_{\mathrm{S_S}} \Big\}.$$

Proposition 2.32. *Let $u \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$.*

1. *If u is a (strong) solution to the problem* (MSP)*, then u is a weak solution to the problem* (MSP)*.*

2. *If u is a weak solution to the problem* (MSP) *such that $\mathrm{Ae}(u)\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{S}}, \mathbb{R}^d)$ and the decomposition $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S_N}} \cup \Gamma_{\mathrm{S_D}} \cup \Gamma_{\mathrm{S_S}}$ of Γ is consistent, then u is a (strong) solution to the problem* (MSP)*.*

Proposition 2.33. *The problem* (MSP) *admits a unique weak solution $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ which is given by*

$$u = \mathrm{prox}_{\mathrm{I}_{\mathcal{K}^1_w(\Omega)}}(F),$$

where $F \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ is the unique solution to the Dirichlet-Neumann problem (DN)*, and $\mathrm{prox}_{\iota_{\mathcal{K}^1_w(\Omega)}}$ stands for the proximal operator associated with the indicator function $\iota_{\mathcal{K}^1_w(\Omega)}$ considered on the Hilbert space $(\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d), \langle \cdot, \cdot \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)})$, where $\iota_{\mathcal{K}^1_w(\Omega)}$ is defined by $\iota_{\mathcal{K}^1_w(\Omega)}(v) := 0$ if $v \in \mathcal{K}^1_w(\Omega)$, and $\iota_{\mathcal{K}^1_w(\Omega)}(v) := +\infty$ otherwise.*

Remark 2.34. *Note that, from Remark 2.10, the unique weak solution $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ to the problem* (MSP) *is also characterized by the projection operator since $\mathrm{prox}_{\iota_{\mathcal{K}^1_w(\Omega)}} = \mathrm{proj}_{\mathcal{K}^1_w(\Omega)}$.*

3 Main results

Let $d \in \{2, 3\}$ and $f \in \mathrm{H}^1(\mathbb{R}^d, \mathbb{R}^d)$. Let Ω_{ref} be a nonempty connected bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary $\Gamma_{\mathrm{ref}} := \partial\Omega_{\mathrm{ref}}$. We assume that $\Gamma_{\mathrm{ref}} = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S_{ref}}}$, where Γ_{D} and $\Gamma_{\mathrm{S_{ref}}}$ are two measurable pairwise disjoint subsets of Γ_{ref}, such that Γ_{D} has a positive measure. We consider the set of admissible shapes $\mathcal{U}_{\mathrm{ref}}$ defined in (1.2). Note that, all shapes in $\mathcal{U}_{\mathrm{ref}}$ have Γ_{D} as common boundary part. We assume that, for all $\Omega \in \mathcal{U}_{\mathrm{ref}}$, Ω is an elastic solid satisfying the linear elastic model with all the same assumptions and notations described at the beginning of Subsection 2.4.

We consider the shape optimization problem (1.1). From Subsection 2.4.2, note that the Signorini energy functional $\mathcal{J}$, given by (1.3), can also be expressed as

$$\mathcal{J}(\Omega) = -\frac{1}{2}\int_{\Omega} \mathrm{Ae}\left(u_{\Omega}\right) : \mathrm{e}\left(u_{\Omega}\right),$$

for all $\Omega \in \mathcal{U}_{\mathrm{ref}}$.

In the whole section let us fix $\Omega_0 \in \mathcal{U}_{\mathrm{ref}}$. Our aim here is to prove that, under appropriate assumptions, the functional $\mathcal{J}$ is *shape differentiable* at Ω_0, in the sense that the map

$$\begin{array}{rcl} \mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d, \mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ \theta & \longmapsto & \mathcal{J}((\mathrm{id}+\theta)(\Omega_0)), \end{array}$$

is Gateaux differentiable at 0, and to give an expression of the Gateaux differential, denoted by $\mathcal{J}'(\Omega_0)$, which is called the shape gradient of $\mathcal{J}$ at Ω_0. To this aim we have to perform the sensitivity analysis of the Signorini problem (SP_{Ω}) with respect to the shape, and then to characterize the material and shape directional derivatives.

This section is separated as follows. In Subsection 3.1, we perturb the Signorini problem with respect to the shape. In Subsection 3.2 we characterize the material derivative as solution to a variational inequality (see Theorem 3.5). Then, with additional regularity assumptions, we characterize the material and shape derivatives as being weak solutions to Signorini problems (see Corollaries 3.6 and 3.8). Finally, in Subsection 3.3, we prove that the Signorini functional $\mathcal{J}$ is shape differentiable at Ω_0, and we provide an expression of its shape gradient (see Theorem 3.10 and Corollary 3.11).

3.1 Setting of the shape perturbation

Consider $\theta \in \mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d, \mathbb{R}^d)$ and, for all $t \geq 0$ sufficiently small such that $\mathrm{id} + t\theta$ is a $\mathcal{C}^2$-diffeomorphism of $\mathbb{R}^d$, consider the shape perturbed Signorini problem given by

$$\left\{ \begin{aligned} -\mathrm{div}(\mathrm{Ae}(u_t)) &= f && \text{in } \Omega_t, \\ u_t &= 0 && \text{on } \Gamma_{\mathrm{D}}, \\ \sigma_{\tau_t}(u_t) &= 0 && \text{on } \Gamma_{\mathrm{S}_t}, \\ u_{t,\mathrm{n}_t} \leq 0,\ \sigma_{\mathrm{n}_t}(u_t) \leq 0 \text{ and } u_{t,\mathrm{n}_t}\sigma_{\mathrm{n}_t}(u_t) &= 0 && \text{on } \Gamma_{\mathrm{S}_t}, \end{aligned} \right. \tag{SP_t}$$

where $\Omega_t := (\mathrm{id}+t\theta)(\Omega_0) \in \mathcal{U}_{\mathrm{ref}}$, $\Gamma_t := (\mathrm{id}+t\theta)(\Gamma_0)$ and n_t is the outward-pointing unit normal vector to Γ_t. From Subsection 2.4.2, there exists a unique solution $u_t \in \mathrm{H}^1(\Omega_t, \mathbb{R}^d)$ to (SP_t) which satisfies

$$\int_{\Omega_t} \mathrm{Ae}(u_t) : \mathrm{e}(v - u_t) \geq \int_{\Omega_t} f \cdot (v - u_t), \qquad \forall v \in \mathcal{K}^1(\Omega_t),$$

where

$$\mathcal{K}^1(\Omega_t) := \left\{ v \in \mathrm{H}^1_{\mathrm{D}}(\Omega_t, \mathbb{R}^d) \mid v_{\mathrm{n}_t} \leq 0 \ a.e. \text{ on } \Gamma_{\mathrm{S}_t} \right\}.$$

Following the usual strategy in shape optimization literature (see, e.g., [5, 20]), using the change of variables $\mathrm{id} + t\theta$ and the equality

$$\mathrm{n}_t \circ (\mathrm{id}+t\theta) = \frac{(\mathrm{I}+t\nabla\theta^\top)^{-1}\mathrm{n}}{\|(\mathrm{I}+t\nabla\theta^\top)^{-1}\mathrm{n}\|},$$

where $\mathrm{n} := \mathrm{n}_0$ (see, e.g., [39, Chapter 2, Proposition 2.48 p.79]) and $||\cdot||$ is the Euclidean norm on $\mathbb{R}^d$, we prove that $\overline{u}_t := u_t \circ (\mathrm{id} + t\theta) \in \mathrm{K}^1_t(\Omega_0) \subset \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$ satisfies

$$\int_{\Omega_0} \mathrm{J}_t \mathrm{A} \left[\nabla \overline{u}_t \left(\mathrm{I} + t\nabla\theta \right)^{-1} \right] : \nabla (v - \overline{u}_t) \left(\mathrm{I} + t\nabla\theta \right)^{-1} \geq \int_{\Omega_0} f_t \mathrm{J}_t \cdot (v - \overline{u}_t), \qquad \forall v \in \mathrm{K}^1_t(\Omega_0), \quad (3.1)$$

where $\mathrm{K}^1_t(\Omega_0) := \{ v \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d) \mid v \cdot (\mathrm{I} + t\nabla\theta^\top)^{-1}\mathrm{n} \leq 0 \ \textit{a.e.} \text{ on } \Gamma_{\mathrm{S}_0} \}$, $f_t := f \circ (\mathrm{id} + t\theta) \in \mathrm{H}^1(\mathbb{R}^d, \mathbb{R}^d)$ and $\mathrm{J}_t := \det(\mathrm{I} + t\nabla\theta) \in \mathrm{L}^\infty(\mathbb{R}^d, \mathbb{R})$ is the Jacobian. Thus, using the characterization of the proximal operator (see Definition 2.9), $\overline{u}_t$ can be expressed as

$$\overline{u}_t = \mathrm{prox}_{\iota_{\mathrm{K}^1_t(\Omega_0)}}(F_t),$$

where $F_t \in \mathrm{H}^1(\Omega_0, \mathbb{R}^d)$ is the unique solution to the parameterized variational equality

$$\int_{\Omega_0} \mathrm{J}_t \mathrm{A} \left[\nabla F_t \left(\mathrm{I} + t\nabla\theta \right)^{-1} \right] : \nabla v \left(\mathrm{I} + t\nabla\theta \right)^{-1} = \int_{\Omega_0} f_t \mathrm{J}_t \cdot v, \qquad \forall v \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d),$$

and $\mathrm{prox}_{\iota_{\mathrm{K}^1_t(\Omega_0)}}$ is the proximal operator associated with the indicator function $\iota_{\mathrm{K}^1_t(\Omega_0)}$ considered on the space $\mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$ endowed with the perturbed scalar product

$$(v_1, v_2) \in \left(\mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d) \right)^2 \mapsto \int_{\Omega_0} \mathrm{J}_t \mathrm{A} \left[\nabla v_1 \left(\mathrm{I} + t\nabla\theta \right)^{-1} \right] : \nabla v_2 \left(\mathrm{I} + t\nabla\theta \right)^{-1} \in \mathbb{R}.$$

Here, the main difficulty is that the indicator function $\iota_{\mathrm{K}^1_t(\Omega_0)}$ depends on the parameter t, thus it would required an extended notion of twice epi-differentiability depending on a parameter in order to apply the Proposition 2.15, like we did in our paper [3]. Nevertheless, it is not necessary since, for all $v \in \mathrm{K}^1_t(\Omega_0)$, one has (similarly to [25, Chapter 5 p.111] and [39, Chapter 4 Section 4.6 p.205])

$$\left(\mathrm{I} + t\nabla\theta \right)^{-1} v \in \mathcal{K}^1(\Omega_0),$$

and reciprocally, for all $\varphi \in \mathcal{K}^1(\Omega_0)$,

$$(\mathrm{I} + t\nabla\theta)\varphi \in \mathrm{K}^1_t(\Omega_0).$$

Thus, from Inequality (3.1), one proves that $\overline{\overline{u}}_t := (\mathrm{I} + t\nabla\theta)^{-1}\overline{u}_t \in \mathcal{K}^1(\Omega_0)$ satisfies

$$\begin{aligned} \int_{\Omega_0} \mathrm{J}_t \mathrm{A} \left[\nabla \left(\left(\mathrm{I} + t\nabla\theta \right) \overline{\overline{u}}_t \right) \left(\mathrm{I} + t\nabla\theta \right)^{-1} \right] &: \nabla \left(\left(\mathrm{I} + t\nabla\theta \right) \left(\varphi - \overline{\overline{u}}_t \right) \right) \left(\mathrm{I} + t\nabla\theta \right)^{-1} \\ &\geq \int_{\Omega_0} \left(\mathrm{I} + t\nabla\theta^\top \right) f_t \mathrm{J}_t \cdot \left(\varphi - \overline{\overline{u}}_t \right), \qquad \forall \varphi \in \mathcal{K}^1(\Omega_0), \quad (3.2) \end{aligned}$$

and can be expressed as

$$\overline{\overline{u}}_t = \mathrm{prox}_{\iota_{\mathcal{K}^1(\Omega_0)}}(G_t),$$

where $G_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$ is the unique solution to the parameterized variational equality

$$\begin{aligned} \int_{\Omega_0} \mathrm{J}_t \mathrm{A} \left[\nabla \left(\left(\mathrm{I} + t\nabla\theta \right) G_t \right) \left(\mathrm{I} + t\nabla\theta \right)^{-1} \right] &: \nabla \left(\left(\mathrm{I} + t\nabla\theta \right) \varphi \right) \left(\mathrm{I} + t\nabla\theta \right)^{-1} \\ &= \int_{\Omega_0} \left(\mathrm{I} + t\nabla\theta^\top \right) f_t \mathrm{J}_t \cdot \varphi, \qquad \forall \varphi \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d), \end{aligned}$$

and $\mathrm{prox}_{\iota_{\mathcal{K}^1(\Omega_0)}}$ is the proximal operator associated with the Signorini indicator function $\iota_{\mathcal{K}^1(\Omega_0)}$ considered on the perturbed Hilbert space $(\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d),\langle\cdot,\cdot\rangle_t)$, where $\langle\cdot,\cdot\rangle_t$ is the scalar product defined by

$$(v_1,v_2)\in\left(\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)\right)^2\mapsto \int_{\Omega_0}\mathrm{J}_t\mathrm{A}\left[\nabla((\mathrm{I}+t\nabla\theta)\,v_1)\,(\mathrm{I}+t\nabla\theta)^{-1}\right]:\ \nabla((\mathrm{I}+t\nabla\theta)\,v_2)\,(\mathrm{I}+t\nabla\theta)^{-1}\in\mathbb{R}.$$

The previous difficulty is solved since the Signorini indicator function $\iota_{\mathcal{K}^1(\Omega_0)}$ does not depend on the parameter $t\geq 0$. Nevertheless, we face here to a perturbed Hilbert space due to the scalar product $\langle\cdot,\cdot\rangle_t$ that is t-dependent, thus we could not apply Theorem 2.15. To overcome this difficulty let us rewrite Inequality (3.2) as (using the equality $\mathrm{B}:\mathrm{CD}=\mathrm{BD}^\top:\mathrm{C}$ for all $\mathrm{B},\mathrm{C},\mathrm{D}\in\mathbb{R}^{d\times d}$)

$$\int_{\Omega_0}\mathrm{J}_t\mathrm{A}\left[\nabla\big((\mathrm{I}+t\nabla\theta)\,\overline{\overline{u}}_t\big)\,(\mathrm{I}+t\nabla\theta)^{-1}\right]\left(\mathrm{I}+t\nabla\theta^\top\right)^{-1}:\nabla\big((\mathrm{I}+t\nabla\theta)\,(\varphi-\overline{\overline{u}}_t)\big)$$
$$\geq\int_{\Omega_0}\left(\mathrm{I}+t\nabla\theta^\top\right)f_t\mathrm{J}_t\cdot\left(\varphi-\overline{\overline{u}}_t\right),\qquad\forall\varphi\in\mathcal{K}^1(\Omega_0),$$

then adding to both members $\left\langle\overline{\overline{u}}_t,\varphi-\overline{\overline{u}}_t\right\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)}$, one deduces

$$\begin{aligned}\left\langle\overline{\overline{u}}_t,\varphi-\overline{\overline{u}}_t\right\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)}\geq&\int_{\Omega_0}\left(\mathrm{I}+t\nabla\theta^\top\right)f_t\mathrm{J}_t\cdot(\varphi-\overline{\overline{u}}_t)\\&-\int_{\Omega_0}\left(\mathrm{J}_t\mathrm{A}\left[\nabla\overline{\overline{u}}_t\,(\mathrm{I}+t\nabla\theta)^{-1}\right]\left(\mathrm{I}+t\nabla\theta^\top\right)^{-1}-\mathrm{A}\nabla\overline{\overline{u}}_t\right):\nabla\big(\varphi-\overline{\overline{u}}_t\big)\\&-t\int_{\Omega_0}\mathrm{J}_t\mathrm{A}\left[\nabla\overline{\overline{u}}_t\,(\mathrm{I}+t\nabla\theta)^{-1}\right]\left(\mathrm{I}+t\nabla\theta^\top\right)^{-1}:\nabla\big(\nabla\theta\,(\varphi-\overline{\overline{u}}_t)\big)\\&-t\int_{\Omega_0}\mathrm{J}_t\mathrm{A}\left[\nabla\big(\nabla\theta\overline{\overline{u}}_t\big)\,(\mathrm{I}+t\nabla\theta)^{-1}\right]\left(\mathrm{I}+t\nabla\theta^\top\right)^{-1}:\nabla\big(\varphi-\overline{\overline{u}}_t\big)\\&-t^2\int_{\Omega_0}\mathrm{J}_t\mathrm{A}\left[\nabla\big(\nabla\theta\overline{\overline{u}}_t\big)\,(\mathrm{I}+t\nabla\theta)^{-1}\right]\left(\mathrm{I}+t\nabla\theta^\top\right)^{-1}:\nabla\big(\nabla\theta\,(\varphi-\overline{\overline{u}}_t)\big),\qquad\forall\varphi\in\mathcal{K}^1(\Omega_0).\end{aligned}$$

Thus $\overline{\overline{u}}_t$ is also expressed as

$$\overline{\overline{u}}_t=\mathrm{prox}_{\iota_{\mathcal{K}^1(\Omega_0)}}(E_t),$$

where $E_t\in\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ stands for the unique solution to the parameterized variational equality

$$\begin{aligned}\langle E_t,\varphi\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)}=&\int_{\Omega_0}\left(\mathrm{I}+t\nabla\theta^\top\right)f_t\mathrm{J}_t\cdot\varphi\\&-\int_{\Omega_0}\left(\mathrm{J}_t\mathrm{A}\left[\nabla\overline{\overline{u}}_t\,(\mathrm{I}+t\nabla\theta)^{-1}\right]\left(\mathrm{I}+t\nabla\theta^\top\right)^{-1}-\mathrm{A}\nabla\overline{\overline{u}}_t\right):\nabla\varphi\\&-t\int_{\Omega_0}\mathrm{J}_t\mathrm{A}\left[\nabla\overline{\overline{u}}_t\,(\mathrm{I}+t\nabla\theta)^{-1}\right]\left(\mathrm{I}+t\nabla\theta^\top\right)^{-1}:\nabla(\nabla\theta\varphi)\\&-t\int_{\Omega_0}\mathrm{J}_t\mathrm{A}\left[\nabla\big(\nabla\theta\overline{\overline{u}}_t\big)\,(\mathrm{I}+t\nabla\theta)^{-1}\right]\left(\mathrm{I}+t\nabla\theta^\top\right)^{-1}:\nabla\varphi\\&-t^2\int_{\Omega_0}\mathrm{J}_t\mathrm{A}\left[\nabla\big(\nabla\theta\overline{\overline{u}}_t\big)\,(\mathrm{I}+t\nabla\theta)^{-1}\right]\left(\mathrm{I}+t\nabla\theta^\top\right)^{-1}:\nabla(\nabla\theta\varphi),\qquad\forall\varphi\in\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d),\end{aligned}\tag{3.3}$$

and where $\operatorname{prox}_{\iota_{\mathcal{K}^1(\Omega_0)}}$ is the proximal operator associated with the Signorini indicator function $\iota_{\mathcal{K}^1(\Omega_0)}$ considered on the Hilbert space $(\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d), \langle\cdot,\cdot\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)})$, which is t-independent.

Now the next step is to derive the differentiability of the map $t \in \mathbb{R}_+ \mapsto E_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ at $t=0$. To this aim let us recall first that (see [20]):

(i) the map $t \in \mathbb{R}_+ \mapsto \mathrm{J}_t \in \mathrm{L}^\infty(\mathbb{R}^d)$ is differentiable at $t=0$ with derivative given by $\operatorname{div}(\theta)$;

(ii) the map $t \in \mathbb{R}_+ \mapsto (\mathrm{I}+t\nabla\theta)^{-1} \in \mathrm{L}^\infty(\mathbb{R}^d,\mathbb{R}^{d\times d})$ is differentiable at $t=0$ with derivative given by $-\nabla\theta$;

(iii) the map $t \in \mathbb{R}_+ \mapsto \left(\mathrm{I}+t\nabla\theta^\top\right)^{-1} \in \mathrm{L}^\infty(\mathbb{R}^d,\mathbb{R}^{d\times d})$ is differentiable at $t=0$ with derivative given by $-\nabla\theta^\top$;

(iv) the map $t \in \mathbb{R}_+ \mapsto \left(\mathrm{I}+t\nabla\theta^\top\right) f_t\mathrm{J}_t \in \mathrm{L}^2(\mathbb{R}^d,\mathbb{R}^d)$ is differentiable at $t=0$ with derivative given by $f\operatorname{div}(\theta)+\nabla f\theta+\nabla\theta^\top f$.

Lemma 3.1. *The map $t \in \mathbb{R}_+ \mapsto E_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ is differentiable at $t=0$ and its derivative, denoted by $E'_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$, is the unique solution to the variational equality given by*

$$\begin{aligned}\langle E'_0,\varphi\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} = &\int_{\Omega_0} \left(f\operatorname{div}(\theta)+\nabla f\theta+\nabla\theta^\top f\right)\cdot\varphi \\ &+\int_{\Omega_0} \left(\left(\mathrm{Ae}(u_0)\right)\nabla\theta^\top + \mathrm{A}\left(\nabla u_0\nabla\theta\right) - \operatorname{div}(\theta)\mathrm{Ae}(u_0)\right):\nabla\varphi \\ &-\int_{\Omega_0}\mathrm{Ae}(u_0):\mathrm{e}\left(\nabla\theta\varphi\right) - \int_{\Omega_0}\mathrm{Ae}\left(\nabla\theta u_0\right):\mathrm{e}\left(\varphi\right), \qquad \forall\varphi\in\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d). \quad (3.4)\end{aligned}$$

Proof. Using the Riesz representation theorem, we denote by $Z \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ the unique solution to the above variational inequality (3.4). From linearity and using differentiability results (i), (ii), (iii), (iv), one gets

$$\begin{aligned}&\left\|\frac{E_t-E_0}{t}-Z\right\|_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} \leq \\ &\qquad C(\Omega_0,d,\mathrm{A},\theta)\Bigg(\left\|\frac{\left(\mathrm{I}+t\nabla\theta^\top\right)f_t\mathrm{J}_t-f}{t}-\left(f\operatorname{div}(\theta)+\nabla f\theta+\nabla\theta^\top f\right)\right\|_{\mathrm{L}^2(\mathbb{R}^d,\mathbb{R}^d)} \\ &\qquad\qquad + \left\|\overline{\overline{u}}_t-u_0\right\|_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} + \frac{o(\theta,t)}{t}\left\|\overline{\overline{u}}_t\right\|_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)}\Bigg),\end{aligned}$$

for all $t\geq 0$ sufficiently small, where $C(\Omega_0,\mathrm{A},\theta,d)>0$ is a constant which depends on $\Omega_0,\mathrm{A},\theta,d$, and where o stands for the standard Bachmann–Landau notation, with $\frac{|o(\theta,t)|}{t}\to 0$ when $t\to 0^+$. Therefore, to conclude the proof, we only need to prove the continuity of the map $t\in\mathbb{R}_+\mapsto \overline{\overline{u}}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ at $t=0$. To this aim let us take $\varphi=u_0$ in the variational formulation of $\overline{\overline{u}}_t$ and $\varphi=\overline{\overline{u}}_t$ in the variational formulation of u_0 to get that

$$\begin{aligned}&\left\|\overline{\overline{u}}_t-u_0\right\|_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} \leq \\ &\qquad C(\Omega_0,\mathrm{A},\theta,d)\left(\left\|\left(\mathrm{I}+t\nabla\theta^\top\right)f_t\mathrm{J}_t-f\right\|_{\mathrm{L}^2(\mathbb{R}^d,\mathbb{R}^d)}+\left\|\overline{\overline{u}}_t\right\|_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)}\left(t+o(\theta,t)\right)\right),\end{aligned}$$

for all $t\geq 0$ sufficiently small. Then, to conclude the proof, we only need to prove that the map $t\in\mathbb{R}_+\mapsto \left\|\overline{\overline{u}}_t\right\|_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)}\in\mathbb{R}$ is bounded for $t\geq 0$ sufficiently small. Let us take $\varphi=0$ in the variational formulation of $\overline{\overline{u}}_t$ to get that

$$\left\|\overline{\overline{u}}_t\right\|^2_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} \leq C(\Omega_0,\mathrm{A},\theta,d)\left\|\left(\mathrm{I}+t\nabla\theta^\top\right)f_t\mathrm{J}_t\right\|_{\mathrm{L}^2(\mathbb{R}^d,\mathbb{R}^d)}\left\|\overline{\overline{u}}_t\right\|_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} + C(\Omega_0,\mathrm{A},\theta,d)\left\|\overline{\overline{u}}_t\right\|^2_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)}\left(t+o(\theta,t)\right),$$

for all $t \geq 0$ sufficiently small. Thus, one deduces

$$\left\|\overline{\overline{u}}_t\right\|_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} \leq \frac{C(\Omega_0,\mathrm{A},\theta,d)(\left\|\left(\mathrm{I}+t\nabla\theta^\top\right)f_t\mathrm{J}_t\right\|_{\mathrm{L}^2(\mathbb{R}^d,\mathbb{R}^d)}}{1-C(\Omega_0,\mathrm{A},\theta,d)\left(t+o(\theta,t)\right)},$$

for all $t \geq 0$ sufficiently small, and using the continuity of the map $t \in \mathbb{R}_+ \mapsto (\mathrm{I}+t\nabla\theta^\top)f_t\mathrm{J}_t \in \mathrm{L}^2(\mathbb{R}^d,\mathbb{R}^d)$ (see (iv)), the proof is complete. □

3.2 Twice epi-differentiability, material and shape directional derivatives

In the previous Subsection 3.1, we have expressed $\overline{\overline{u}}_t = \mathrm{prox}_{\iota_{\mathcal{K}^1(\Omega_0)}}(E_t)$ and characterized in Lemma 3.1 the derivative of the map $t \in \mathbb{R}_+ \mapsto E_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ at $t=0$. Thus, we will use Proposition 2.15 to differentiate the map $t \in \mathbb{R}_+ \mapsto \overline{\overline{u}}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ at $t=0$ and then to deduce the material directional derivative. To this aim the twice epi-differentiability of the Signorini indicator function $\iota_{\mathcal{K}^1(\Omega_0)}$ has to be investigated. Hence, from Lemma 2.14, we have to prove that the set $\mathcal{K}^1(\Omega_0)$ is polyhedric. This result has already been proved in [25, Lemma 5.2.9 p.116] involving some concepts from convex analysis and capacity theory, reminded in Subsections 2.1 and 2.2.

Lemma 3.2. *The nonempty closed convex subset $\mathcal{K}^1(\Omega_0)$ of $\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ is polyhedric at $u_0 \in \mathcal{K}^1(\Omega_0)$ for $E_0-u_0 \in \mathrm{N}_{\mathcal{K}^1(\Omega_0)}(u_0)$, and one has*

$$\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap \left(\mathbb{R}\left(E_0-u_0\right)\right)^\perp = \left\{\varphi \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d) \mid \varphi\cdot\mathrm{n} \leq 0 \text{ q.e. on } \Gamma^{u_{0\mathrm{n}}}_{\mathrm{S}_0} \text{ and } \langle E_0-u_0,\varphi\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)}=0\right\},$$

where $\Gamma^{u_{0\mathrm{n}}}_{\mathrm{S}_0} := \{s \in \Gamma_{\mathrm{S}_0} \mid u_{0\mathrm{n}}(s)=0\}$.

Using the previous lemma and Lemma 2.14, one can deduce the following proposition.

Proposition 3.3. *The Signorini indicator function $\iota_{\mathcal{K}^1(\Omega_0)}$ is twice epi-differentiable at $u_0 \in \mathcal{K}^1(\Omega_0)$ for $E_0-u_0 \in \mathrm{N}_{\mathcal{K}^1(\Omega_0)}(u_0)$ and*

$$\mathrm{d}^2_e\iota_{\mathcal{K}^1(\Omega_0)}(u_0|E_0-u_0) = \iota_{\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0)\cap(\mathbb{R}(E_0-u_0))^\perp}.$$

The twice epi-differentiability of the Signorini indicator function allows us to prove that the map $t \in \mathbb{R}_+ \mapsto \overline{\overline{u}}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ is differentiable at $t=0$.

Theorem 3.4. *Consider the framework of Subsection 3.1. Then the map $t \in \mathbb{R}_+ \mapsto \overline{\overline{u}}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)$ is differentiable at $t=0$ and its derivative, denoted by $\overline{\overline{u}}'_0 \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}(E_0-u_0))^\perp$, is the unique solution to the variational inequality*

$$\left\langle\overline{\overline{u}}'_0,\varphi-\overline{\overline{u}}'_0\right\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} \geq -\int_{\Omega_0}\mathrm{div}\left(\mathrm{div}\left(\mathrm{Ae}(u_0)\right)\theta^\top\right)\cdot\left(\varphi-\overline{\overline{u}}'_0\right) + \int_{\Omega_0}\left(\left(\mathrm{Ae}(u_0)\right)\nabla\theta^\top + \mathrm{A}\left(\nabla u_0\nabla\theta\right) - \mathrm{Ae}\left(\nabla\theta u_0\right) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right) : \nabla\left(\varphi-\overline{\overline{u}}'_0\right)$$

$$- \left\langle \mathrm{Ae}(u_0)\mathrm{n}, \nabla\theta\left(\varphi - \overline{\overline{u}}_0'\right)\right\rangle_{\mathrm{H}^{-1/2}(\Gamma_0,\mathbb{R}^d)\times\mathrm{H}^{1/2}(\Gamma_0,\mathbb{R}^d)}, \quad (3.5)$$

for all $\varphi \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^{\perp}$.

Proof. For all $t \geq 0$, $\overline{\overline{u}}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$ is given by

$$\overline{\overline{u}}_t = \mathrm{prox}_{\iota_{\mathcal{K}^1(\Omega_0)}}(E_t),$$

where $E_t \in \mathrm{H}^1(\Omega_0, \mathbb{R}^d)$ stands for the unique solution to the parameterized variational equality (3.3). Moreover the map $t \in \mathbb{R}_+ \mapsto E_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$ is differentiable at $t = 0$ with its derivative $E_0' \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$ solution to the variational inequality (3.4). Therefore, from Proposition 3.3 one can apply Proposition 2.15 to deduce the differentiability of the map $t \in \mathbb{R}_+ \mapsto \overline{\overline{u}}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$, with its derivative $\overline{\overline{u}}_0' \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$ given by

$$\overline{\overline{u}}_0' = \mathrm{prox}_{\iota_{\mathrm{T}_{\mathcal{K}^1(\Omega_0)}(u_0)\cap(\mathbb{R}(E_0-u_0))^{\perp}}}(E_0'),$$

which, from definition of the proximal operator (see Definition 2.9), leads to

$$\left\langle E_0' - \overline{\overline{u}}_0', \varphi - \overline{\overline{u}}_0' \right\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} \leq 0,$$

for all $\varphi \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^{\perp}$. Using the variational equality satisfied by E_0' (see (3.4)), one gets

$$\begin{aligned}\left\langle \overline{\overline{u}}_0', \varphi - \overline{\overline{u}}_0' \right\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} \geq{}& \int_{\Omega_0} \left(f\mathrm{div}(\theta) + \nabla f\theta + \nabla\theta^{\top} f\right) \cdot \left(\varphi - \overline{\overline{u}}_0'\right) \\ &+ \int_{\Omega_0} \left((\mathrm{Ae}(u_0))\,\nabla\theta^{\top} + \mathrm{A}\,(\nabla u_0 \nabla\theta) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right) : \nabla\left(\varphi - \overline{\overline{u}}_0'\right) \\ &- \int_{\Omega_0} \mathrm{Ae}(u_0) : \mathrm{e}\left(\nabla\theta\left(\varphi - \overline{\overline{u}}_0'\right)\right) - \int_{\Omega_0} \mathrm{Ae}\,(\nabla\theta u_0) : \mathrm{e}\left(\varphi - \overline{\overline{u}}_0'\right),\end{aligned}$$

for all $\varphi \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^{\perp}$, which is also (see the notations introduced at the beginning of Subsection 2.4)

$$\begin{aligned}\left\langle \overline{\overline{u}}_0', \varphi - \overline{\overline{u}}_0' \right\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} \geq{}& \int_{\Omega_0} \mathrm{div}(f\theta^{\top}) \cdot \left(\varphi - \overline{\overline{u}}_0'\right) + \int_{\Omega_0} f \cdot \nabla\theta\left(\varphi - \overline{\overline{u}}_0'\right) \\ &+ \int_{\Omega_0} \left((\mathrm{Ae}(u_0))\,\nabla\theta^{\top} + \mathrm{A}\,(\nabla u_0 \nabla\theta) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right) : \nabla\left(\varphi - \overline{\overline{u}}_0'\right) \\ &- \int_{\Omega_0} \mathrm{Ae}(u_0) : \mathrm{e}\left(\nabla\theta\left(\varphi - \overline{\overline{u}}_0'\right)\right) - \int_{\Omega_0} \mathrm{Ae}\,(\nabla\theta u_0) : \mathrm{e}\left(\varphi - \overline{\overline{u}}_0'\right),\end{aligned}$$

for all $\varphi \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^{\perp}$. Using the divergence formula (see Proposition 2.22) and the equality $\mathrm{div}\,(\mathrm{Ae}(u_0)) = -f$ in $\mathrm{H}^1(\Omega_0, \mathbb{R}^d)$, we obtain the result. □

Since $\overline{u}_t = (\mathrm{I} + t\nabla\theta)\,\overline{\overline{u}}_t$, it is possible now to state and prove the first main result of this paper that characterizes the material directional derivative.

Theorem 3.5 (Material directional derivative)**.** *Consider the framework of Theorem 3.4. Then the map* $t \in \mathbb{R}_+ \mapsto \overline{u}_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$ *is differentiable at* $t = 0$ *and its derivative (that is, the material directional derivative), denoted by* $\overline{u}'_0 \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^{\perp} + \nabla\theta u_0$*, is the unique solution to the variational inequality*

$$\begin{aligned}\langle \overline{u}'_0, \varphi - \overline{u}'_0 \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} \geq &-\int_{\Omega_0} \mathrm{div}\left(\mathrm{div}\left(\mathrm{Ae}(u_0)\right)\theta^{\top}\right)\cdot(\varphi - \overline{u}'_0) \\ &+ \int_{\Omega_0} \left((\mathrm{Ae}(u_0))\,\nabla\theta^{\top} + \mathrm{A}\left(\nabla u_0 \nabla\theta\right) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right) : \nabla(\varphi - \overline{u}'_0) \\ &- \langle \mathrm{Ae}(u_0)\mathrm{n}, \nabla\theta\,(\varphi - \overline{u}'_0)\rangle_{\mathrm{H}^{-1/2}(\Gamma_0,\mathbb{R}^d)\times\mathrm{H}^{1/2}(\Gamma_0,\mathbb{R}^d)}, \end{aligned} \tag{3.6}$$

for all $\varphi \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^{\perp} + \nabla\theta u_0$*, where*

$$\begin{aligned}&\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^{\perp} + \nabla\theta u_0 = \\ &\quad \left\{ \varphi \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d) \mid \varphi\cdot\mathrm{n} \leq (\nabla\theta u_0)\cdot\mathrm{n} \text{ q.e. on } \Gamma^{u_0\mathrm{n}}_{\mathrm{S}_0} \text{ and } \langle E_0 - u_0, \varphi - \nabla\theta u_0\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} = 0 \right\}.\end{aligned}$$

Proof. Since $\overline{u}_t = (\mathrm{I} + t\nabla\theta)\overline{\overline{u}}_t$, then one deduces from Theorem 3.4 that $\overline{u}'_0 = \overline{\overline{u}}'_0 + \nabla\theta u_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$. Moreover, from the variational formulation of $\overline{\overline{u}}'_0$ (see (3.5)), one deduces

$$\begin{aligned}\langle \overline{u}'_0 - \nabla\theta u_0, \varphi + \nabla\theta u_0 - \overline{u}'_0 \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} \geq &-\int_{\Omega_0} \mathrm{div}\left(\mathrm{div}\left(\mathrm{Ae}(u_0)\right)\theta^{\top}\right)\cdot(\varphi + \nabla\theta u_0 - \overline{u}'_0) \\ &+ \int_{\Omega_0} \left((\mathrm{Ae}(u_0))\,\nabla\theta^{\top} + \mathrm{A}\left(\nabla u_0 \nabla\theta\right) - \mathrm{Ae}\left(\nabla\theta u_0\right) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right) : \nabla(\varphi + \nabla\theta u_0 - \overline{u}'_0) \\ &- \langle \mathrm{Ae}(u_0)\mathrm{n}, \nabla\theta\,(\varphi + \nabla\theta u_0 - \overline{u}'_0)\rangle_{\mathrm{H}^{-1/2}(\Gamma_0,\mathbb{R}^d)\times\mathrm{H}^{1/2}(\Gamma_0,\mathbb{R}^d)}, \end{aligned}$$

for all $\varphi \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^{\perp}$, and this is also

$$\begin{aligned}\langle \overline{u}'_0 - \nabla\theta u_0, \varphi - \overline{u}'_0 \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} \geq &-\int_{\Omega_0} \mathrm{div}\left(\mathrm{div}\left(\mathrm{Ae}(u_0)\right)\theta^{\top}\right)\cdot(\varphi - \overline{u}'_0) \\ &+ \int_{\Omega_0} \left((\mathrm{Ae}(u_0))\,\nabla\theta^{\top} + \mathrm{A}\left(\nabla u_0 \nabla\theta\right) - \mathrm{Ae}\left(\nabla\theta u_0\right) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right) : \nabla(\varphi - \overline{u}'_0) \\ &- \langle \mathrm{Ae}(u_0)\mathrm{n}, \nabla\theta\,(\varphi - \overline{u}'_0)\rangle_{\mathrm{H}^{-1/2}(\Gamma_0,\mathbb{R}^d)\times\mathrm{H}^{1/2}(\Gamma_0,\mathbb{R}^d)}, \end{aligned}$$

for all $\varphi \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^{\perp} + \nabla\theta u_0$, which concludes the proof. □

In [9], [25, Chapter 5 Section 5.2 p.111] and [39, Chapter 4 Section 4.6 p.205], the authors get the same result using the conical differentiability of the projection operator. Since $\mathcal{K}^1(\Omega_0)$ is polyhedric at $u_0 \in \mathcal{K}^1(\Omega_0)$ for $E_0 - u_0 \in \mathrm{N}_{\mathcal{K}^1(\Omega_0)}(u_0)$, then from Mignot's theorem (see [26]) the projection operator on $\mathcal{K}^1(\Omega_0)$ is conically differentiable at u_0 for $E_0 - u_0$, and its conical derivative is given by $\mathrm{proj}_{\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0)\cap(\mathbb{R}(E_0-u_0))^{\perp}}(E'_0)$, which is exactly $\mathrm{prox}_{\iota_{\mathrm{T}_{\mathcal{K}^1(\Omega_0)}(u_0)\cap(\mathbb{R}(E_0-u_0))^{\perp}}}(E'_0)$. Nevertheless, to the best of our knowledge, no one notices that it was possible to improve this result under additional assumptions, in order to characterize the material derivative as weak solution to a boundary value problem. Indeed, as mentioned in [9, 21], it is possible to replace *q.e.* in the set $\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^{\perp}$ by *a.e.* under some hypotheses, like, for instance, if $\overline{\Gamma^{u_0\mathrm{n}}_{\mathrm{S}_0}} = \overline{\mathrm{int}_{\Gamma_{\mathrm{S}_0}}(\Gamma^{u_0\mathrm{n}}_{\mathrm{S}_0})}$. Moreover, if we assume that the decomposition $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}_0}$ of Γ_0 is consistent (see Definition 2.31 with $\Gamma_{\mathrm{S}_{\mathrm{S}}} := \Gamma_{\mathrm{S}_0}$ and $w = 0$) and some regularity assumptions on u_0 and θ, then they allow us to characterize the material derivative as weak solution to a Signorini problem.

Corollary 3.6. *Consider the framework of Theorem 3.5 with the additional assumptions that the decomposition* $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}_0}$ *of* Γ_0 *is consistent,* $u_0 \in \mathrm{H}^3(\Omega_0, \mathbb{R}^d)$ *and* $\Gamma_{\mathrm{S}_0}^{\mathrm{u}_{0\mathrm{n}}} = \overline{\mathrm{int}_{\Gamma_{\mathrm{S}_0}}(\Gamma_{\mathrm{S}_0}^{\mathrm{u}_{0\mathrm{n}}})}$*. Then the material directional derivative* $\overline{u}_0' \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^{\perp} + \nabla\theta u_0$ *is the unique weak solution to the Signorini problem*

$$
\begin{cases}
-\mathrm{div}(\mathrm{Ae}(\overline{u}_0')) = -\mathrm{div}(\mathrm{Ae}(\nabla u_0\theta)) & \text{in } \Omega_0,\\
\overline{u}_0' = 0 & \text{on } \Gamma_{\mathrm{D}},\\
\sigma_\tau(\overline{u}_0') = h^m(\theta)_\tau & \text{on } \Gamma_{\mathrm{S}_0},\\
\sigma_{\mathrm{n}}(\overline{u}_0') = h^m(\theta)_{\mathrm{n}} & \text{on } \Gamma_{\mathrm{S}_{0,\mathrm{N}}^{u_{0\mathrm{n}}}},\\
\overline{u}_{0\mathrm{n}}' = (\nabla\theta u_0)_{\mathrm{n}} & \text{on } \Gamma_{\mathrm{S}_{0,\mathrm{D}}^{u_{0\mathrm{n}}}},\\
\overline{u}_{0\mathrm{n}}' \leq (\nabla\theta u_0)_{\mathrm{n}}\,, \sigma_{\mathrm{n}}(\overline{u}_0') \leq h^m(\theta)_{\mathrm{n}} \text{ and } \left(\overline{u}_{0\mathrm{n}}' - (\nabla\theta u_0)_{\mathrm{n}}\right)\left(\sigma_{\mathrm{n}}(\overline{u}_0') - h^m(\theta)_{\mathrm{n}}\right) = 0 & \text{on } \Gamma_{\mathrm{S}_{0,\mathrm{S}}^{u_{0\mathrm{n}}}},
\end{cases}
$$

where $h^m(\theta) := ((\mathrm{Ae}(u_0))\nabla\theta^{\top} + \mathrm{A}(\nabla u_0\nabla\theta) - \sigma_{\mathrm{n}}(u_0)(\mathrm{div}(\theta)\mathrm{I} + \nabla\theta^{\top}))\mathrm{n} \in \mathrm{L}^2(\Gamma_0, \mathbb{R}^d)$*, and* Γ_{S_0} *is decomposed, up to a null set, as* $\Gamma_{\mathrm{S}_{0,\mathrm{N}}^{u_{0\mathrm{n}}}} \cup \Gamma_{\mathrm{S}_{0,\mathrm{D}}^{u_{0\mathrm{n}}}} \cup \Gamma_{\mathrm{S}_{0,\mathrm{S}}^{u_{0\mathrm{n}}}}$*, where*

$$
\begin{aligned}
\Gamma_{\mathrm{S}_{0,\mathrm{N}}^{u_{0\mathrm{n}}}} &:= \{s \in \Gamma_{\mathrm{S}_0} \mid u_{0\mathrm{n}}(s) \neq 0\}\,,\\
\Gamma_{\mathrm{S}_{0,\mathrm{D}}^{u_{0\mathrm{n}}}} &:= \{s \in \Gamma_{\mathrm{S}_0} \mid u_{0\mathrm{n}}(s) = 0 \text{ and } \sigma_{\mathrm{n}}(u_0)(s) < 0\}\,,\\
\Gamma_{\mathrm{S}_{0,\mathrm{S}}^{u_{0\mathrm{n}}}} &:= \{s \in \Gamma_{\mathrm{S}_0} \mid u_{0\mathrm{n}}(s) = 0 \text{ and } \sigma_{\mathrm{n}}(u_0)(s) = 0\}\,.
\end{aligned}
$$

Proof. Since $u_0 \in \mathrm{H}^2(\Omega_0, \mathbb{R}^d)$ and $\theta \in \mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d, \mathbb{R}^d)$, one deduces that

$$
\mathrm{div}\left((\mathrm{Ae}(u_0))\,\nabla\theta^{\top} + \mathrm{A}\,(\nabla u_0\nabla\theta) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right) \in \mathrm{L}^2(\Omega_0, \mathbb{R}^d).
$$

Moreover, since $\mathrm{Ae}(u_0)\mathrm{n} \in \mathrm{L}^2(\Gamma_0, \mathbb{R}^d)$ and that the decomposition $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}_0}$ of Γ_0 is consistent, then u_0 is a (strong) solution to the Signorini problem (SP_t) for the parameter $t = 0$ (see Proposition 2.32). Thus $\sigma_\tau(u_0) = 0$ *a.e.* on Γ_{S_0}, and using the divergence formula (see Proposition 2.22) in Inequality (3.6), we get that

$$
\begin{aligned}
\langle \overline{u}_0', \varphi - \overline{u}_0' \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} \geq{}& \int_{\Gamma_{\mathrm{S}_0}} h^m \cdot (\varphi - \overline{u}_0')\\
&- \int_{\Omega_0} \mathrm{div}\left(\mathrm{div}(\mathrm{Ae}(u_0))\theta^{\top} + (\mathrm{Ae}(u_0))\,\nabla\theta^{\top} + \mathrm{A}\,(\nabla u_0\nabla\theta) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right) \cdot (\varphi - \overline{u}_0')\,, \quad (3.7)
\end{aligned}
$$

for all $\varphi \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^{\perp} + \nabla\theta u_0$. Furthermore, one has $\mathrm{div}\,(\mathrm{Ae}\,(\nabla u_0\theta)) \in \mathrm{L}^2(\Omega_0, \mathbb{R}^d)$ from $u_0 \in \mathrm{H}^3(\Omega_0, \mathbb{R}^d)$. Thus, using the equality

$$
\mathrm{div}\,(\mathrm{Ae}\,(\nabla u_0\theta)) = \mathrm{div}\left(\mathrm{div}(\mathrm{Ae}(u_0))\theta^{\top} + (\mathrm{Ae}(u_0))\,\nabla\theta^{\top} + \mathrm{A}\,(\nabla u_0\nabla\theta) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)\right),
$$

in $\mathrm{L}^2(\Omega_0, \mathbb{R}^d)$, it follows that

$$
\langle \overline{u}_0', \varphi - \overline{u}_0' \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} \geq -\int_{\Omega_0} \mathrm{div}\,(\mathrm{Ae}\,(\nabla u_0\theta)) \cdot (\varphi - \overline{u}_0') + \int_{\Gamma_{\mathrm{S}_0}} h^m \cdot (\varphi - \overline{u}_0')\,,
$$

for all $\varphi \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^{\perp} + \nabla\theta u_0$. Moreover, since

$$
\mathcal{H} := \left\{ v \cdot \mathrm{n} \in \mathrm{H}^{1/2}(\Gamma_{\mathrm{S}}, \mathbb{R}) \mid v \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d) \right\},
$$

is a Dirichlet space (see Example 2.21), then, for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$, $v \cdot \mathrm{n} \in \mathrm{H}^{1/2}(\Gamma_{\mathrm{S}}, \mathbb{R})$ admits a unique quasi-continuous representative for the *q.e* equivalence relation (see Proposition 2.19), thus it follows that (see [9, Remark 3.12 p.13] for details)

$$\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^\perp$$
$$= \left\{ \varphi \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d) \mid \varphi \cdot \mathrm{n} \leq 0 \ \textit{a.e.} \text{ on } \Gamma^{u_{0\mathrm{n}}}_{\mathrm{S}_0} \text{ and } \langle E_0 - u_0, \varphi \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} = 0 \right\}.$$

Furthermore, since u_0 is a (strong) solution, it follows from the Signorini unilateral conditions that

$$\langle E_0 - u_0, v \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} = \int_{\Gamma_0} \mathrm{Ae}(E_0 - u_0) \cdot v = \int_{\Gamma_{\mathrm{S}_0}} \sigma_{\mathrm{n}}(u_0) v_{\mathrm{n}} = \int_{\Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{D}}}} \sigma_{\mathrm{n}}(u_0) v_{\mathrm{n}} = 0,$$

for all $v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^\perp$, and since $\sigma_{\mathrm{n}}(u_0) v_{\mathrm{n}} \leq 0$ *a.e.* on $\Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{D}}}$ and $\sigma_{\mathrm{n}}(u_0) < 0$ *a.e.* on $\Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{D}}}$, one deduces that $v_{\mathrm{n}} = 0$ *a.e.* on $\Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{D}}}$, for all $v \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^\perp$, which concludes the proof from Subsection 2.4.2. □

Remark 3.7. Note that, from the proof of Corollary 3.6, one can get, under the weaker assumption $u_0 \in \mathrm{H}^2(\Omega_0, \mathbb{R}^d)$, that the material directional derivative $\overline{u}'_0$ is the solution to the variational inequality (3.7) which is, from Subsection 2.4.2, the weak formulation of a Signorini problem, with the source term given by $-\mathrm{div}(\mathrm{div}(\mathrm{Ae}(u_0))\theta^\top + (\mathrm{Ae}(u_0))\nabla\theta^\top + \mathrm{A}(\nabla u_0 \nabla\theta) - \mathrm{div}(\theta)\mathrm{Ae}(u_0)) \in \mathrm{L}^2(\Omega_0, \mathbb{R}^d)$. It is important to note that, to the best of our knowledge, there is no regularity result for the solution to the Signorini problem with respect to the data. Obtaining this regularity result in our case is a highly nontrivial work and is not the main focus of this paper. However, we can mention the works [34, 35] which deal with regularity results for variational inequalities concerning the Stokes equations, and also [7, Chapter 1, Theorem I.10 p.43] which concerns a scalar Signorini-type problem.

Thanks to Corollary 3.6, we are now in a position to characterize the shape directional derivative.

Corollary 3.8 (Shape directional derivative)**.** *Consider the framework of Corollary 3.6 with the additional assumption that Γ_0 is of class $\mathcal{C}^3$. Then the shape directional derivative, defined by $u'_0 := \overline{u}'_0 - \nabla u_0 \theta \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^\perp + \nabla\theta u_0 - \nabla u_0 \theta$ is the unique weak solution to the Signorini problem*

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(u'_0)) = 0 & \text{in } \Omega_0, \\ u'_0 = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \sigma_\tau(u'_0) = h^s(\theta)_\tau & \text{on } \Gamma_{\mathrm{S}_0}, \\ \sigma_{\mathrm{n}}(u'_0) = h^s(\theta)_{\mathrm{n}} & \text{on } \Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{N}}}, \\ u'_{0\mathrm{n}} = W(\theta)_{\mathrm{n}} & \text{on } \Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{D}}}, \\ u'_{0\mathrm{n}} \leq W(\theta)_{\mathrm{n}}, \sigma_{\mathrm{n}}(u'_0) \leq h^s(\theta)_{\mathrm{n}} \text{ and } (u'_{0\mathrm{n}} - W(\theta)_{\mathrm{n}})\,(\sigma_{\mathrm{n}}(u'_0) - h^s(\theta)_{\mathrm{n}}) = 0 & \text{on } \Gamma_{\mathrm{S}^{u_{0\mathrm{n}}}_{0,\mathrm{S}}}, \end{cases}$$

where $W(\theta) := (\nabla\theta u_0) - (\nabla u_0 \theta) \in \mathrm{H}^{1/2}(\Gamma_0, \mathbb{R}^d)$,

$$h^s(\theta) := \theta \cdot \mathrm{n}\,(\partial_{\mathrm{n}}(\mathrm{Ae}(u_0)\mathrm{n}) - \partial_{\mathrm{n}}(\mathrm{Ae}(u_0))\,\mathrm{n}) + \mathrm{Ae}(u_0)\nabla_\tau(\theta \cdot \mathrm{n})$$
$$- \nabla(\mathrm{Ae}(u_0)\mathrm{n})\theta - \sigma_{\mathrm{n}}(u_0)\left(\mathrm{div}_\tau(\theta)\mathrm{I} + \nabla\theta^\top\right)\mathrm{n} \in \mathrm{L}^2(\Gamma_0, \mathbb{R}^d),$$

and where $\partial_{\mathrm{n}}(\mathrm{Ae}(u_0)\mathrm{n}) := \nabla(\mathrm{Ae}(u_0)\mathrm{n})\mathrm{n}$ stands for the normal derivative of $\mathrm{Ae}(u_0)\mathrm{n}$, and $\partial_{\mathrm{n}}(\mathrm{Ae}(u_0))$ is the matrix whose the i-th line is the vector $\partial_{\mathrm{n}}(\mathrm{Ae}(u_0)_i) := \nabla(\mathrm{Ae}(u_0)_i)\mathrm{n}$, where $\mathrm{Ae}(u_0)_i$ is the i-th line of the matrix $\mathrm{Ae}(u_0)$, for all $i \in [[1, d]]$.

Proof. Since $u'_0 := \overline{u}'_0 - \nabla u_0 \theta$, one deduces from the weak formulation of $\overline{u}'_0$ and using the divergence formula that,

$$\langle u_0', \varphi - u_0' \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} \geq$$
$$\int_{\Omega_0} \big(\mathrm{div}\,(\mathrm{Ae}(u_0))\,\theta^\top + (\mathrm{Ae}(u_0))\,\nabla\theta^\top + \mathrm{A}\,(\nabla u_0 \nabla\theta) - \mathrm{Ae}\,(\nabla u_0 \theta)\big) : \nabla(\varphi - u_0')$$
$$- \int_{\Omega_0} \mathrm{div}(\theta)\mathrm{Ae}(u_0) : \mathrm{e}\,(\varphi - u_0') - \int_{\Gamma_{\mathrm{S}_0}} \mathrm{Ae}(u_0)\mathrm{n} \cdot \nabla\theta\,(\varphi - u_0') + \int_{\Gamma_{\mathrm{S}_0}} (\theta\cdot\mathrm{n})\,f\cdot(\varphi - u_0')\,,$$

for all $\varphi \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^\perp + \nabla\theta u_0 - \nabla u_0 \theta$. Moreover, one has

$$\int_{\Omega_0} \mathrm{div}\,(\mathrm{Ae}(u_0))\,\theta^\top : \nabla v = \int_{\Omega_0} \mathrm{div}\,(\mathrm{Ae}(u_0)) \cdot \nabla v\theta = -\int_{\Omega_0} \mathrm{Ae}(u_0) : \nabla(\nabla v \theta) + \int_{\Gamma_0} \mathrm{Ae}(u_0)\mathrm{n}\cdot\nabla v\theta,$$

and also

$$-\int_{\Omega_0} \mathrm{div}(\theta)\mathrm{Ae}(u_0) : \mathrm{e}(v) = \int_{\Omega_0} \theta\cdot\nabla(\mathrm{Ae}(u_0) : \mathrm{e}(v)) - \int_{\Gamma_0} \theta\cdot\mathrm{n}\,(\mathrm{Ae}(u_0) : \mathrm{e}(v))\,,$$

for all $v \in \mathcal{C}^\infty(\overline{\Omega_0}, \mathbb{R}^d)$. Therefore, using the equality

$$\big((\mathrm{Ae}(u_0))\,\nabla\theta^\top + \mathrm{A}\,(\nabla u_0\nabla\theta) - \mathrm{Ae}\,(\nabla u_0\theta)\big) : \nabla v + \theta\cdot\nabla(\mathrm{Ae}(u_0) : \mathrm{e}(v)) - \mathrm{Ae}(u_0) : \nabla(\nabla v\theta) = 0,$$

a.e. on Ω_0, one deduces using the divergence formula that

$$\int_{\Omega_0} \big(\mathrm{div}\,(\mathrm{Ae}(u_0))\,\theta^\top + (\mathrm{Ae}(u_0))\,\nabla\theta^\top + \mathrm{A}\,(\nabla u_0\nabla\theta) - \mathrm{Ae}\,(\nabla u_0\theta)\big) : \nabla v$$
$$- \int_{\Omega_0} \mathrm{div}(\theta)\mathrm{Ae}(u_0) : \mathrm{e}\,(v) - \int_{\Gamma_0} \nabla\theta^\top\,(\mathrm{Ae}(u_0)\mathrm{n})\cdot v + \int_{\Gamma_0} (\theta\cdot\mathrm{n})\,f\cdot v$$
$$= \int_{\Gamma_0} \theta\cdot\mathrm{n}\,(-\mathrm{Ae}(u_0) : \mathrm{e}(v) + f\cdot v) + \nabla v^\top(\mathrm{Ae}(u_0)\mathrm{n})\cdot\theta - \nabla\theta^\top(\mathrm{Ae}(u_0)\mathrm{n})\cdot v,$$

for all $v \in \mathcal{C}^\infty(\overline{\Omega_0}, \mathbb{R}^d)$. Furthermore, since Γ_0 is of class $\mathcal{C}^3$ and $u_0 \in \mathrm{H}^3(\Omega_0, \mathbb{R}^d)$, $\mathrm{Ae}(u_0)\mathrm{n}$ can be extended into a function defined in Ω_0 such that $\mathrm{Ae}(u_0)\mathrm{n} \in \mathrm{H}^2(\Omega_0, \mathbb{R}^d)$. Thus, it holds that $\mathrm{Ae}(u_0)\mathrm{n}\cdot v \in \mathrm{W}^{2,1}(\Omega_0, \mathbb{R}^d)$, and one can use Proposition 2.23 to get that

$$\int_{\Gamma_0} \theta\cdot\mathrm{n}\,(-\mathrm{Ae}(u_0) : \mathrm{e}(v) + f\cdot v) + \nabla v^\top(\mathrm{Ae}(u_0)\mathrm{n})\cdot\theta - \nabla\theta^\top(\mathrm{Ae}(u_0)\mathrm{n})\cdot v$$
$$= \int_{\Gamma_0} \theta\cdot\mathrm{n}\,(-\mathrm{Ae}(u_0) : \mathrm{e}(v) + f\cdot v + \partial_{\mathrm{n}}\,(\mathrm{Ae}(u_0)\mathrm{n}\cdot v) + H\mathrm{Ae}(u_0)\mathrm{n}\cdot v)$$
$$- \int_{\Gamma_0} \big(\nabla(\mathrm{Ae}(u_0)\mathrm{n})\theta + \nabla\theta^\top(\mathrm{Ae}(u_0)\mathrm{n}) + \mathrm{div}_\tau(\theta)\mathrm{Ae}(u_0)\mathrm{n}\big)\cdot v,$$

for all $v \in \mathcal{C}^\infty(\overline{\Omega_0}, \mathbb{R}^d)$. Moreover, since $-\mathrm{div}(\mathrm{Ae}(u_0)) = f \in \mathrm{H}^1(\Omega_0, \mathbb{R}^d)$, one deduces from Proposition 2.24 that

$$\int_{\Gamma_0} \theta\cdot\mathrm{n}\,(f + H\mathrm{Ae}(u_0)\mathrm{n})\cdot v = \int_{\Gamma_0} \mathrm{Ae}(u_0) : \nabla_\tau\,(v\,(\theta\cdot\mathrm{n})) - (\theta\cdot\mathrm{n})\,\partial_{\mathrm{n}}\,(\mathrm{Ae}(u_0))\,\mathrm{n}\cdot v,$$

for all $v \in \mathcal{C}^\infty(\overline{\Omega_0}, \mathbb{R}^d)$. Therefore, using the following equalities

$$\mathrm{Ae}(u_0) : \nabla_\tau\,(v\,(\theta\cdot\mathrm{n})) = \theta\cdot\mathrm{n}\,(\mathrm{Ae}(u_0) : \nabla_\tau v) + \mathrm{Ae}(u_0)\nabla_\tau\,(\theta\cdot\mathrm{n})\cdot v,\ \ \textit{a.e.}\text{ on }\Gamma_0,$$

and

$$\mathrm{Ae}(u_0):\nabla_\tau v=\mathrm{Ae}(u_0):e(v)-\nabla v^\top(\mathrm{Ae}(u_0)\mathrm{n})\cdot\mathrm{n}\ \ a.e \text{ on } \Gamma_0,$$

one gets

$$\begin{aligned}\int_{\Gamma_0}&\theta\cdot\mathrm{n}\left(-\mathrm{Ae}(u_0):\mathrm{e}(v)+f\cdot v+\partial_\mathrm{n}\left(\mathrm{Ae}(u_0)\mathrm{n}\cdot v\right)+H\mathrm{Ae}(u_0)\mathrm{n}\cdot v\right)\\&\qquad-\int_{\Gamma_0}\left(\nabla(\mathrm{Ae}(u_0)\mathrm{n})\theta+\nabla\theta^\top(\mathrm{Ae}(u_0)\mathrm{n})+\mathrm{div}_\tau(\theta)\mathrm{Ae}(u_0)\mathrm{n}\right)\cdot v\\&=\int_{\Gamma_0}\left(\theta\cdot\mathrm{n}\left(\partial_\mathrm{n}\left(\mathrm{Ae}(u_0)\mathrm{n}\right)-\partial_\mathrm{n}\left(\mathrm{Ae}(u_0)\right)\mathrm{n}\right)+\mathrm{Ae}(u_0)\nabla_\tau\left(\theta\cdot\mathrm{n}\right)\right)\cdot v\\&\qquad\qquad-\int_{\Gamma_0}\left(\nabla(\mathrm{Ae}(u_0)\mathrm{n})\theta+\left(\mathrm{div}_\tau(\theta)\mathrm{I}+\nabla\theta^\top\right)\mathrm{Ae}(u_0)\mathrm{n}\right)\cdot v,\end{aligned}$$

for all $v\in\mathcal{C}^\infty(\overline{\Omega_0},\mathbb{R}^d)$. Thus,

$$\begin{aligned}\int_{\Omega_0}&\left(\mathrm{div}\left(\mathrm{Ae}(u_0)\right)\theta^\top+\left(\mathrm{Ae}(u_0)\right)\nabla\theta^\top+\mathrm{A}\left(\nabla u_0\nabla\theta\right)-\mathrm{Ae}\left(\nabla u_0\theta\right)\right):\nabla v\\&\qquad-\int_{\Omega_0}\mathrm{div}(\theta)\mathrm{Ae}(u_0):\mathrm{e}\left(v\right)-\int_{\Gamma_0}\nabla\theta^\top\left(\mathrm{Ae}(u_0)\mathrm{n}\right)\cdot v+\int_{\Gamma_0}\left(\theta\cdot\mathrm{n}\right)f\cdot v\\&=\int_{\Gamma_0}\left(\theta\cdot\mathrm{n}\left(\partial_\mathrm{n}\left(\mathrm{Ae}(u_0)\mathrm{n}\right)-\partial_\mathrm{n}\left(\mathrm{Ae}(u_0)\right)\mathrm{n}\right)+\mathrm{Ae}(u_0)\nabla_\tau\left(\theta\cdot\mathrm{n}\right)\right)\cdot v\\&\qquad\qquad-\int_{\Gamma_0}\left(\nabla(\mathrm{Ae}(u_0)\mathrm{n})\theta+\left(\mathrm{div}_\tau(\theta)\mathrm{I}+\nabla\theta^\top\right)\mathrm{Ae}(u_0)\mathrm{n}\right)\cdot v,\end{aligned}$$

for all $v\in\mathcal{C}^\infty(\overline{\Omega_0},\mathbb{R}^d)$, and one deduces by density of $\mathcal{C}^\infty(\overline{\Omega_0},\mathbb{R}^d)$ in $\mathrm{H}^1(\Omega_0,\mathbb{R}^d)$ that

$$\begin{aligned}\langle u_0',\varphi-u_0'\rangle_{\mathrm{H}^1_\mathrm{D}(\Omega_0,\mathbb{R}^d)}\geq&\int_{\Gamma_{\mathrm{S}_0}}\left(\theta\cdot\mathrm{n}\left(\partial_\mathrm{n}\left(\mathrm{Ae}(u_0)\mathrm{n}\right)-\partial_\mathrm{n}\left(\mathrm{Ae}(u_0)\right)\mathrm{n}\right)+\mathrm{Ae}(u_0)\nabla_\tau\left(\theta\cdot\mathrm{n}\right)\right)\cdot\left(\varphi-u_0'\right)\\&-\int_{\Gamma_{\mathrm{S}_0}}\left(\nabla(\mathrm{Ae}(u_0)\mathrm{n})\theta+\sigma_\mathrm{n}(u_0)\left(\mathrm{div}_\tau(\theta)\mathrm{I}+\nabla\theta^\top\right)\mathrm{n}\right)\cdot\left(\varphi-u_0'\right),\end{aligned}$$

for all $\varphi\in\mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0)\cap\left(\mathbb{R}\left(E_0-u_0\right)\right)^\perp+\nabla\theta u_0-\nabla u_0\theta$, which concludes the proof from Subsection 2.4.2. □

Remark 3.9. Note that $\overline{u}_0'$ and u_0' are not linear with respect to the direction θ. This nonlinearity is standard in shape optimization for variational inequalities (see, e.g., [3, 21] or [39, Section 4]), and justifies the names of material and shape *directional* derivatives.

3.3 Shape gradient of the Signorini energy functional

Thanks to the characterization of the material and shape directional derivatives obtained in the previous section, we are now in a position to prove the shape differentiability of the Signorini energy functional (1.3).

Theorem 3.10. *Consider the framework of Theorem 3.5. Then the Signorini energy functional $\mathcal{J}$ admits a shape gradient at Ω_0 in the direction $\theta\in\mathcal{C}^{2,\infty}_\mathrm{D}(\mathbb{R}^d,\mathbb{R}^d)$ given by*

$$\mathcal{J}'(\Omega_0)(\theta) = \int_{\Omega_0} \operatorname{div}(\theta) \frac{\mathrm{Ae}(u_0) : \mathrm{e}(u_0)}{2} - \int_{\Omega_0} \operatorname{div}(\mathrm{Ae}(u_0)) \cdot \nabla u_0 \theta - \int_{\Omega_0} \mathrm{Ae}(u_0) : \nabla u_0 \nabla \theta$$
$$- \int_{\Gamma_{\mathrm{S}_0}} \theta \cdot \mathrm{n}\,(f \cdot u_0) + \langle \mathrm{Ae}(u_0)\mathrm{n}, \nabla\theta u_0 \rangle_{\mathrm{H}^{-1/2}(\Gamma_0,\mathbb{R}^d)\times \mathrm{H}^{1/2}(\Gamma_0,\mathbb{R}^d)}.$$

Proof. By following the usual strategy developed in the shape optimization literature (see, e.g., [5, 20]) to compute the shape gradient of $\mathcal{J}$ at Ω_0 in the direction $\theta \in \mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d, \mathbb{R}^d)$, one gets

$$\mathcal{J}'(\Omega_0)(\theta) = -\frac{1}{2} \int_{\Omega_0} \operatorname{div}(\theta) \mathrm{Ae}(u_0) : \mathrm{e}(u_0) + \int_{\Omega_0} \mathrm{Ae}(u_0) : \nabla u_0 \nabla\theta - \langle \overline{u}'_0, u_0 \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)}.$$

Moreover, one has

$$\langle \overline{u}'_0, u_0 \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} = \left\langle \overline{\overline{u}}'_0, u_0 \right\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} + \langle \nabla\theta u_0, u_0 \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)},$$

and, since $\overline{\overline{u}}'_0 \pm u_0 \in \mathrm{T}_{\mathrm{N}_{\mathcal{K}^1(\Omega_0)}}(u_0) \cap (\mathbb{R}\,(E_0 - u_0))^\perp$, one deduces from the variational formulation of $\overline{\overline{u}}'_0$ (see Inequality (3.5)) and the divergence formula that

$$\langle \overline{u}'_0, u_0 \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} = \int_{\Omega_0} \big(\operatorname{div}(\mathrm{Ae}(u_0))\, \theta^\top + (\mathrm{Ae}(u_0))\, \nabla\theta^\top + \mathrm{A}\,(\nabla u_0 \nabla\theta) - \operatorname{div}(\theta) \mathrm{Ae}(u_0) \big) : \nabla u_0$$
$$+ \int_{\Gamma_{\mathrm{S}_0}} \theta \cdot \mathrm{n}(f \cdot u_0) - \langle \mathrm{Ae}(u_0)\mathrm{n}, \nabla\theta u_0 \rangle_{\mathrm{H}^{-1/2}(\Gamma_0,\mathbb{R}^d)\times \mathrm{H}^{1/2}(\Gamma_0,\mathbb{R}^d)}.$$

Then, using the equality $\operatorname{div}(\mathrm{Ae}(u_0))\, \theta^\top : \nabla u_0 = \operatorname{div}(\mathrm{Ae}(u_0)) \cdot \nabla u_0 \theta$ *a.e.* on Ω_0, one concludes the proof. □

As we did for the material directional derivative, the presentation of Theorem 3.10 can be improved under additional assumption.

Corollary 3.11. *Consider the framework of Theorem 3.10 and assume that $u_0 \in \mathrm{H}^2(\Omega_0, \mathbb{R}^d)$. Then the Signorini energy functional $\mathcal{J}$ admits a shape gradient at Ω_0 in the direction $\theta \in \mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d, \mathbb{R}^d)$ given by*

$$\mathcal{J}'(\Omega_0)(\theta) = \int_{\Gamma_{\mathrm{S}_0}} \left(\theta \cdot \mathrm{n} \left(\frac{\mathrm{Ae}(u_0) : \mathrm{e}(u_0)}{2} - f \cdot u_0 \right) + \mathrm{Ae}(u_0)\mathrm{n} \cdot (\nabla\theta u_0 - \nabla u_0 \theta) \right).$$

Proof. Let $\theta \in \mathcal{C}^{2,\infty}_{\mathrm{D}}(\mathbb{R}^d, \mathbb{R}^d)$. Since $u_0 \in \mathrm{H}^2(\Omega_0, \mathbb{R}^d)$, it follows from Theorem 3.10 that

$$\mathcal{J}'(\Omega_0)(\theta) = -\frac{1}{2} \int_{\Omega_0} \theta \cdot \nabla(\mathrm{Ae}(u_0) : \mathrm{e}(u_0)) + \int_{\Gamma_0} \theta \cdot \mathrm{n} \frac{\mathrm{Ae}(u_0) : \mathrm{e}(u_0)}{2} + \int_{\Omega_0} \mathrm{Ae}(u_0) : \mathrm{e}(\nabla u_0 \theta)$$
$$- \int_{\Gamma_0} \mathrm{Ae}(u_0)\,\mathrm{n} \cdot \nabla u_0 \theta - \int_{\Omega_0} \mathrm{Ae}(u_0) : \nabla u_0 \nabla\theta - \int_{\Gamma_{\mathrm{S}_0}} \theta \cdot \mathrm{n}\,(f \cdot u_0) + \int_{\Gamma_{\mathrm{S}_0}} \mathrm{Ae}(u_0)\mathrm{n} \cdot \nabla\theta u_0.$$

Moreover, since

$$\mathrm{Ae}(u_0) : \mathrm{e}(\nabla u_0 \theta) = \mathrm{Ae}(u_0) : \nabla u_0 \nabla\theta + \frac{1}{2} \theta \cdot \nabla(\mathrm{Ae}(u_0) : \mathrm{e}(u_0)) \ \textit{a.e.} \text{ on } \Omega_0,$$

one deduces

$$\mathcal{J}'(\Omega_0)(\theta) = \int_{\Gamma_0} \theta \cdot \mathrm{n} \left(\frac{\mathrm{Ae}(u_0) : \mathrm{e}(u_0)}{2} \right) - \int_{\Gamma_0} \mathrm{Ae}(u_0)\mathrm{n} \cdot \nabla u_0 \theta - \int_{\Gamma_{\mathrm{S}_0}} \theta \cdot \mathrm{n}(f \cdot u_0) + \int_{\Gamma_0} \mathrm{Ae}(u_0)\mathrm{n} \cdot \nabla\theta u_0,$$

which completes the proof since $\theta = 0$ on Γ_{D}. □

Remark 3.12. Consider the framework of Theorem 3.10. It is interesting to note that the scalar product $\langle \overline{u}_0', u_0 \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)}$ is linear with respect to the direction θ, while $\overline{u}_0'$ is not. This leads to an expression of the shape gradient $\mathcal{J}'(\Omega_0)(\theta)$ in Theorem 3.10 that is linear with respect to the direction θ, thus to the shape differentiability of the Signorini energy functional $\mathcal{J}$ at Ω_0. Note that the shape gradient $\mathcal{J}'(\Omega_0)(\theta)$ depends only on u_0 (and not on u_0'), therefore its expression is explicit with respect to the direction θ, and there is no need to introduce any adjoint problem to perform numerical simulations. Nevertheless, for other cost functionals, some difficulties can appear to correctly define an adjoint problem due to nonlinearities in shape gradients, and may constitute an interesting area for future researches. This was already noticed in our paper [3, Remark 3.15 p.21] in the context of a scalar Tresca friction problem.

4 Numerical simulations

In this section we numerically solve an example of the shape optimization problem (1.1) in the two-dimensional case $d = 2$, by making use of our theoretical results obtained in Section 3. The numerical simulations have been performed using Freefem++ software [18] with P1-finite elements and standard affine mesh. We could use the expression of the shape gradient of $\mathcal{J}$ obtained in Theorem 3.10 but, in order to simplify the computations, we chose to use the expression provided in Corollary 3.11 under the additional assumption $u_0 \in \mathrm{H}^2(\Omega_0, \mathbb{R}^d)$ that we assumed to be true at each iteration.

4.1 Numerical methodology

Consider an initial shape $\Omega_0 \in \mathcal{U}_{\mathrm{ref}}$. Note that Corollary 3.11 allows to exhibit a descent direction θ_0 of the Signorini energy functional $\mathcal{J}$ at Ω_0 by finding the solution θ_0 to the variational equality

$$\langle \theta_0, \theta \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} = -\mathcal{J}'(\Omega_0)(\theta), \qquad \forall \theta \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d),$$

since it satisfies $\mathcal{J}'(\Omega_0)(\theta_0) = -\left\|\theta_0\right\|^2_{\mathrm{H}^1_{\mathrm{D}}(\Omega_0,\mathbb{R}^d)} \leq 0$.

In order to numerically solve the shape optimization problem (1.1) on a given example, we have to deal with the volume constraint $|\Omega| = |\Omega_{\mathrm{ref}}| > 0$. To this aim, the Uzawa algorithm (see, e.g., [5, Chapter 3 p.64]) is used, and one refers to [3, Section 4] for methodological details.

Let us mention that the Signorini problem is numerically solved using the Nitsche's method (see, e.g., [10, 11, 29]). In a nutshell, the solution $u_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega_0, \mathbb{R}^d)$ is approximated by $u_0^h \in \mathbb{V}^h$ which is the solution to the Nitsche's formulation

$$\int_{\Omega_0} \mathrm{A}\mathrm{e}(u_0^h) : \mathrm{e}(v^h) - \gamma \int_{\Gamma_{\mathrm{S}_0}} \sigma_{\mathrm{n}}(u_0^h)\sigma_{\mathrm{n}}(v^h) + \frac{1}{\gamma} \int_{\Gamma_{\mathrm{S}_0}} \left[{u_0}_{\mathrm{n}}^h - \gamma\sigma_{\mathrm{n}}(u_0^h)\right]_+ \left[v_{\mathrm{n}}^h - \gamma\sigma_{\mathrm{n}}(v^h)\right]$$
$$= \int_{\Omega_0} f \cdot v^h, \qquad \forall v^h \in \mathbb{V}^h,$$

where $\mathbb{V}^h$ is the classical P1-finite elements space whose elements are null on Γ_{D} (see [11] for numerical analysis details). We also precise that, for all $i \in \mathbb{N}^*$, the difference between the Signorini energy functional $\mathcal{J}$ at the iteration $20 \times i$ and at the iteration $20 \times (i-1)$ is computed. The smallness of this difference is used as a stopping criterion for the algorithm.

4.2 Two-dimensional example and numerical results

In this subsection, let $d = 2$ and $f \in \mathrm{H}^1(\mathbb{R}^2, \mathbb{R}^2)$ given by

$$f: \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & f(x,y) := \left(\frac{1}{2}\exp(x^2)\eta(x,y) \quad 0\right), \end{array}$$

where $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^2, \mathbb{R})$ is a cut-off function chosen appropriately so that f satisfies the assumptions of the present paper. The reference shape Ω_{ref} is the unit disk of $\mathbb{R}^2$, and the fixed part Γ_{D} is given by

$$\Gamma_{\mathrm{D}} = \left\{ (\cos\alpha, \sin\alpha) \in \Gamma_{\mathrm{ref}} \mid \alpha \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right] \cup \left[\frac{7\pi}{6}, \frac{11\pi}{6}\right] \right\},$$

(see Figure 1). The volume constraint is $|\Omega_{\mathrm{ref}}| = \pi$ and the initial shape is $\Omega_0 := \Omega_{\mathrm{ref}}$. We assume that all shapes in $\mathcal{U}_{\mathrm{ref}}$ are isotropic, which means that their mechanical properties are identical in all directions. In that case, for all $\Omega \in \mathcal{U}_{\mathrm{ref}}$, the Cauchy stress tensor is given, for all $v \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, by

$$\sigma(v) = 2\mu\mathrm{e}(v) + \lambda\mathrm{tr}\left(\mathrm{e}(v)\right)\mathrm{I},$$

where $\mathrm{tr}\left(\mathrm{e}(v)\right)$ is the trace of the matrix $\mathrm{e}(v)$, and $\mu \geq 0, \lambda \geq 0$ are Lamé parameters (see, e.g., [36]). In what follows, we consider $\mu = 0.3846$, $\lambda = 0.5769$, and one presents the numerical results obtained for this two-dimensional example using the methodology described in Subsection 4.1.

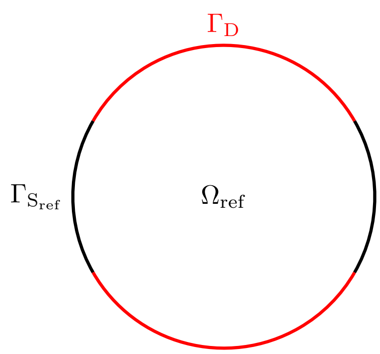

Figure 1: Unit disk Ω_{ref} and its boundary $\Gamma_{\mathrm{ref}} = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S_{ref}}}$.

In Figure 2 is represented the initial shape (left) and the shape which solves Problem (1.1) (right). On top are the vector values of the solution u to the Signorini problem (SP_Ω). Note that the black boundary shows where $\sigma_{\mathrm{n}}(u) = 0$, while the yellow boundary shows where $u_{\mathrm{n}} = 0$. At the bottom is shown the values of the integrand of $\mathcal{J}$. It seems that the area where the integrand of $\mathcal{J}$ is the lowest (in orange) has been shifted to the left by "pushing" the left boundary (which corresponds to the part where there is no compressive stress), while in return, the right boundary (which corresponds to the contact part) has been pulled.

Figure 3 shows the values of $\mathcal{J}$ (left) and the volume of the shape (right) with respect to the iteration. We observe that $\mathcal{J}$ is lower at the final shape, than at the initial shape, with some oscillations due to the Lagrange multiplier in order to satisfy the volume constraint.

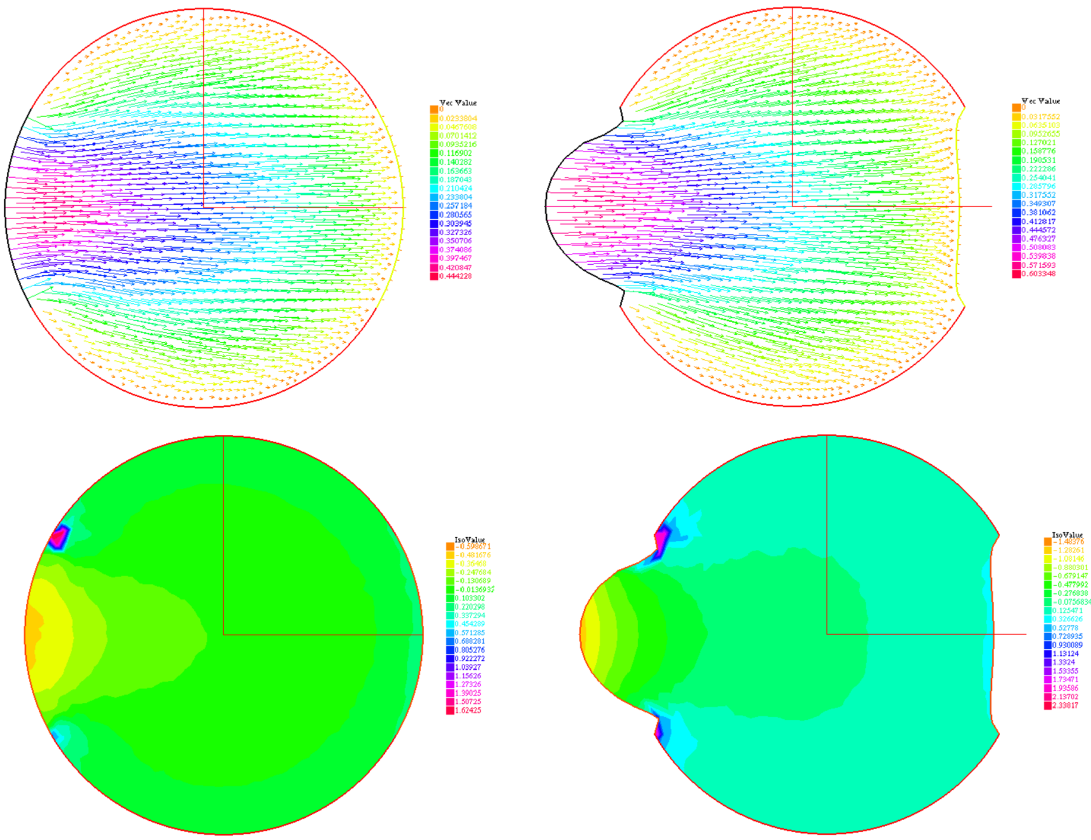

Figure 2: Initial shape (left) and the shape minimizing $\mathcal{J}$ (right), under the volume constraint $|\Omega_{\text{ref}}| = \pi$. On top is shown the vector values of the Signorini solution, while at bottom is shown the values of the integrand of $\mathcal{J}$.

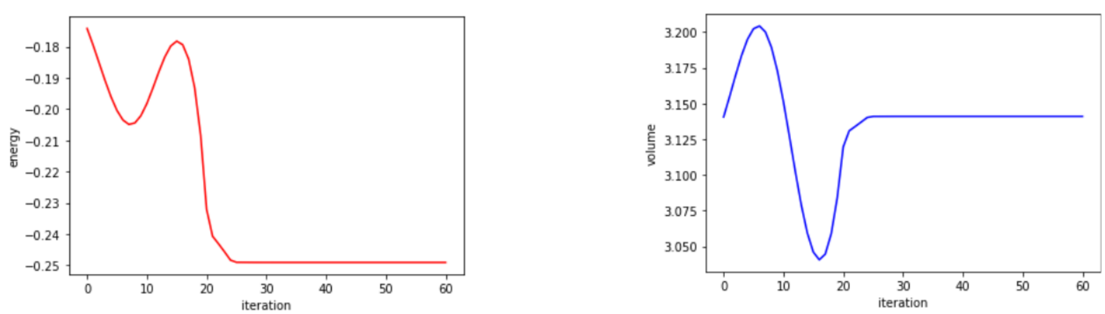

Figure 3: Energy (left) and the volume (right) with respect to the iteration.

References

[1] S. Adly and L. Bourdin. Sensitivity analysis of variational inequalities via twice epi-differentiability and proto-differentiability of the proximity operator. *SIAM Journal on Opti-*

mization, 28(2):1699–1725, 2018.

[2] S. Adly, L. Bourdin, and F. Caubet. Sensitivity analysis of a Tresca-type problem leads to Signorini's conditions. *ESAIM: COCV*, 28(29), 2022.

[3] S. Adly, L. Bourdin, F. Caubet, and A. Jacob de Cordemoy. Shape optimization for variational inequalities: the scalar Tresca friction problem. Submitted. In revision, 2023.

[4] G. Allaire. *Analyse numerique et optimisation.* Mathématiques et Applications. Éditions de l'École Polytechnique., 2007.

[5] G. Allaire. *Conception optimale de structures.* Mathématiques et Applications. Springer-Verlag Berlin Heidelberg, 2007.

[6] G. Allaire, F. Jouve, and A. Maury. Shape optimisation with the level set method for contact problems in linearised elasticity. *The SMAI journal of computational mathematics*, 3:249–292, 2017.

[7] H. Brezis. Problèmes unilatéraux. *J. Math. Pures Appl*, 51:1–168, 1972.

[8] H. Brezis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, volume 5 of *North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).* North-Holland Publishing Co., Amsterdam, 1973.

[9] B. Chaudet-Dumas and J. Deteix. Shape derivatives for an augmented lagrangian formulation of elastic contact problems. *ESAIM: Control, Optimisation and Calculus of Variations*, 27(14):23, 2021.

[10] F. Chouly. An adaptation of nitsches method to the tresca friction problem. *Journal of Mathematical Analysis and Applications*, 411(1):329–339, 2014.

[11] F. Chouly and P. Hild. A Nitsche-based method for unilateral contact problems: numerical analysis. *SIAM Journal on Numerical Analysis*, 51(2):p. 1295–1307, 2013.

[12] J. Deny. Théorie de la capacité dans les espaces fonctionnels. *Séminaire Brelot-Choquet-Deny. Théorie du potentiel*, 9, 1964-1965. talk:1.

[13] C. N. Do. Generalized second-order derivatives of convex functions in reflexive banach spaces. *Transactions of the American Mathematical Society*, 334(1):281–301, 1992.

[14] G. Duvaut and J.-L. Lions. *Les inéquations en mécanique et en physique.* Travaux et Recherches Mathématiques, No. 21. Dunod, Paris, 1972.

[15] P. Fulmański, A. Laurain, J.-F. Scheid, and J. Sokoł owski. A level set method in shape and topology optimization for variational inequalities. *Int. J. Appl. Math. Comput. Sci.*, 17(3):413–430, 2007.

[16] A. Haraux. How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities. *Journal of the Mathematical Society of Japan*, 29(4):615 – 631, 1977.

[17] J. Haslinger and A. Klarbring. Shape optimization in unilateral contact problems using generalized reciprocal energy as objective functional. *Nonlinear Anal.*, 21(11):815–834, 1993.

[18] F. Hecht. New development in freefem++. *J. Numer. Math.*, 20(3-4):251–265, 2012.

[19] A. Henrot, I. Mazari, and Y. Privat. Shape optimization of a Dirichlet type energy for semilinear elliptic partial differential equations. working paper or preprint, Apr. 2020.

[20] A. Henrot and M. Pierre. *Shape Variation and Optimization : a Geometrical Analysis*. Tracts in Mathematics Vol. 28. European Mathematical Society, 2018.

[21] M. Hintermüller and A. Laurain. Optimal shape design subject to elliptic variational inequalities. *SIAM Journal on Control and Optimization*, 49(3):1015–1047, 2011.

[22] N. Kikuchi and J. T. Oden. *Contact Problems in Elasticity*. Society for Industrial and Applied Mathematics, 1988.

[23] D. Luft, V. H. Schulz, and K. Welker. Efficient techniques for shape optimization with variational inequalities using adjoints. *SIAM Journal on Optimization*, 30(3):1922–1953, 2020.

[24] K. Malanowski. Remarks on Differentiability of Metric Projections onto Cones of Nonnegative Functions. *Journal of Convex Analysis*, 10(1):285—294, 2003.

[25] A. Maury. *Shape optimization for contact and plasticity problems thanks to the level set method*. Thèse, Université Pierre et Marie Curie - Paris VI, 2016.

[26] F. Mignot. Contrôle dans les inéquations variationelles elliptiques. *Journal of Functional Analysis*, 22(2):130–185, 1976.

[27] G. J. Minty. Monotone (nonlinear) operators in Hilbert space. *Duke Mathematical Journal*, 29(3):341–346, 1962.

[28] J. J. Moreau. Proximité et dualité dans un espace hilbertien. *Bulletin de la Société Mathématique de France*, 93:273–299, 1965.

[29] J. Nitsche. Über ein variationsprinzip zur lösung von dirichlet-problemen bei verwendung von teilräumen, die keinen randbedingungen unterworfen sind. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 36:9–15, 1971.

[30] R. T. Rockafellar. On the maximal monotonicity of subdifferential mappings. *Pacific Journal of Mathematics*, 33(1):209 – 216, 1970.

[31] R. T. Rockafellar. Maximal monotone relations and the second derivatives of nonsmooth functions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 2(3):167–184, 1985.

[32] R. T. Rockafellar. Generalized second derivatives of convex functions and saddle functions. *Trans. Amer. Math. Soc.*, 322(1):51–77, 1990.

[33] R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis*, volume 317 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg, 1998.

[34] N. Saito. Regularity of solutions to some variational inequalities for the Stokes equations. Number 1181, pages 182–193. 2001. Variational problems and related topics (Japanese) (Kyoto, 2000).

[35] N. Saito and H. Fujita. Regularity of solutions to the Stokes equations under a certain nonlinear boundary condition. In *The Navier-Stokes equations: theory and numerical methods (Varenna, 2000)*, volume 223 of *Lecture Notes in Pure and Appl. Math.*, pages 73–86. Dekker, New York, 2002.

[36] J. Salençon. *Handbook of continuum mechanics (general concepts, thermoelasticity)*. Springer-Verlag, Berlin, 2001.

[37] A. Signorini. Sopra alcune questioni di statica dei sistemi continui. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, Ser. 2, 2(2):231–251, 1933.

[38] A. Signorini. Questioni di elasticità non linearizzata e semilinearizzata. *Rend. Mat. Appl., V. Ser.*, 18:95–139, 1959.

[39] J. Sokołowski and J. Zolésio. *Introduction to shape optimization*, volume 16 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1992.

[40] B. Velichkov. *Existence and regularity results for some shape optimization problems*. Theses, Université de Grenoble, Nov. 2013.