# Robust obstacle reconstruction in an elastic medium

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**Abstract**

We study the inverse problem of the reconstruction of an obstacle in an elastic media from boundary measurements. We assume that the data is noised and that a statistical model for the data is at hand. We propose and study a reconstruction algorithm based a weighted combination of the first two moments of the Kohn-Vogelius criterion. By numerical results in dimension two, the applicability and feasibility of our approach is demonstrated.

## 1 Introduction

This work is devoted to the mathematical study of an inverse problem in linear elasticity, namely the identification of an unknown inclusion in an elastic body from measurements performed on the boundary of the object. This type of inverse problem has many practical applications: for example, in field of geophysical exploration and medical imaging. The problem under consideration is a special case of the reconstruction problem and is severely ill-posed.

Among the numerous approaches in order the solve this type of problem, we focuss on the shape optimization point of view introduced by Roche & Sokolowski, [15], then developped in a large literature (see for example Caubet, Dambrine, and Harbrecht [2], Afraites, Dambrine, and Kateb [1], Meftahi & Zolesio [13], and Lazarev & Rudoy, [12]). Inverse problems are then solved by minimizing an appropriate objective function by the gradient method. This is a standard strategy in handling a mismatch between model predictions and real measurements. To this end, it is necessary to calculate the gradient of the objective function with respect to the shape variations.

The main feature of this work is that we allow noise in the measurements of displacements at the border. We assume that the noise on measurement has a special structure and our objective in this article is to take advantage of properties of the noise to construct a deterministic formulation which incorporates this knowledge. We assume that the measured flux is given as a random field that models the measurement errors. We then aim at minimizing a combination of the expectation and the variance of the Kohn-Vogelius functional as presented approach in [3, 4]. As we will show, both quantities can easily be computed via deterministic quantities. The associated shape gradients can likewise be deterministically computed.

There are at least two canonical choice of objective to study the problem. The first was developed by Kohn & Vogelius [11], Roche & Sokolowski [15], and by Eppler & Harbrecht [6]. It is based on the analysis of the optimization problem for the Kohn-Vogelius functional. The second approach is the consideration of least-squares tracking functionals as developed by Afraites, Dambrine, and Kateb [1], and by Rudoy [16].

The remarkable property of the the Kohn-Vogelius functional is that the expression for its shape gradient does not contain the shape derivative of solutions to the governing equations. In this work, we follow this point of view. The shape derivative of the Kohn-Vogelius functional can be represented as integral of quadratic form, depending on the gradient of the governing equations, over the boundary of inclusion. This integral is well defined only for sufficiently smooth inclusions. Hence the problem of calculation of shape derivatives for this functional is, in some sense, ill-posed and might never be resolved in general case. This fact was mentioned by Eppler, Harbrecht, and Schneider [7], and first analyzed by Eppler & Harbrecht [6]. Moreover because of the flatness of the reconstruction objective in the vicinity of its minimizers analyzed in the works [6, 1], the standard gradient descent method no longer shows a satisfactory behavior. In this work, we propose to combine Nesterov inertial scheme to accelerate the descent and a regularization by projection on a finite dimensional space the admissible class of domains.

The organization of the paper is as follows. In Section 2, we formulate the identification elastic inclusion problem as a shape optimization problem for the Kohn-Vogelius functional. We prove the existence of solutions to the corresponding optimization problem in the class of inclusions satisfying the Feireisl type condition, [8]. This restriction is natural and provides the compactness of the set of inclusions in the Hausdorff metric.

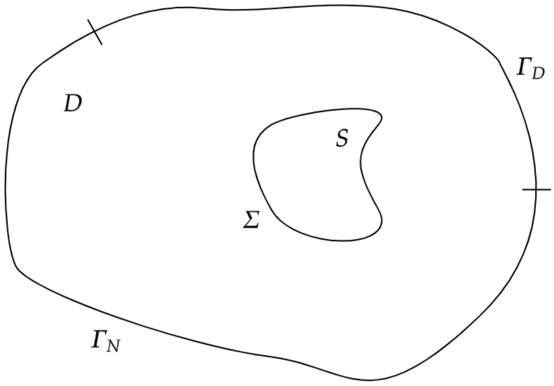
Section 3 is devoted to a detailed calculation of shape gradient of the Kohn-Vogelius functional. In particular, we also derive the expression for the shape derivative of solutions to the governing elastic equations. Finally, we give two representations for the shape gradient of the Kohn-Vogelius functional in the distributed and singular forms.

Section 4 is concerned with the discretization of the shape optimization problem. We assume that the elastic inclusion is a star-shaped domain which enables us to approximate it by a finite Fourier series. We present a numerical method for solving the problem. A distinctive feature of this methods is the systematic use of Nesterov's adaptive restart algorithm. In Section 4, we also present numerical results. Finally, in Section 5, we state concluding remarks. Technical proofs are gathered in Section 6.

## 2 Problem formulation

The problem is formulated as follows to reconstruct an unknown inclusion in an elastic body from additional Dirichlet data containing noise. There is existence but we can say nothing about uniqueness. Moreover, the problems of reconstructing the inclusions are severely unstable. Especially, in the case with the noised measured data.

**Notations.** Let $D \subset \mathbb{R}^2$ be a simply connected domain with Lipschitz boundary $\Gamma$, which is divided into two subsets $\Gamma_D$ and $\Gamma_N$ such that $\Gamma = \overline{\Gamma}_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$ and the measure of $\Gamma_D$ is non negative. Moreover it is assumed that there is unknown inclusion $S \subset D$ with regular boundary $\Sigma$.




In what follows, we use the following notation for the derivatives of the vector field in $\mathbb{R}^2$ and the deformation tensor $\varepsilon(\phi)$:

$$\frac{\partial \phi_i}{\partial x_j} \equiv \phi_{i,j}, \quad \varepsilon_{ij}(\phi) = \frac{1}{2}(\phi_{i,j} + \phi_{j,i}) = \frac{1}{2}\big(\nabla\phi + (\nabla\phi)^\top\big),$$

for normal derivative $\partial_n u$ and jump $[u]$:

$$\partial_n u = \nabla u \cdot n, \quad [u] = \lim_{\epsilon\to 0}(u(x+\epsilon n(x)) - u(x - \epsilon n(x))),$$

where $n$ represents the exterior unit normal vector on $\partial D$.

We assume that the region $D$ is filled with inhomogeneous elastic material, the state of which is completely characterized by the vector field of displacements $u = (u_1, u_2) : D \to \mathbb{R}^2$. Material properties are fully characterized by fourth order tensor $\mathbb{C} = \{c_{ijkl}\}$, $i, j, k, l = 1, 2$. The inhomogeneity of the body is that the inclusion characteristics differ from the other part. This is expressed in the following property of $\mathbb{C}$

$$\mathbb{C}(x) = \begin{cases} \mathbb{C}^1(x) & \text{if } x \in D \setminus \overline{S}, \\ \mathbb{C}^2(x) & \text{if } x \in \overline{S}. \end{cases}$$

We assume that $\mathbb{C}$ satisfies the following symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{klij},$$

and the ellipticity condition

$$c^{-1}|\xi|^2 \;\le\; c_{klij}\,\xi_{ij}\,\xi_{kl} \;\le\; c|\xi|^2. \tag{2.1}$$

Also it should be noted that we consider case when $c^{1,2}_{ijkl}$ is positive constants. The stress tensor is determined by the relations

$$\sigma(\phi) = \mathbb{C}\varepsilon(\phi), \;\; \sigma_{ij}(\phi) = c_{jikl}\;\varepsilon_{kl}(\phi).$$

Here $f \in H^{-\frac{1}{2}}(\Gamma_N)$ - force applied to the unsecured part of the boundary.

**Direct problem** Differential equations and boundary conditions (2.2) - (2.4) describe the equilibrium of an elastic body fixed on a part of the boundary under the action of an external surface force with an elastic inclusion. In mathematical terms, for some fixed domain $D$ and function $f \in H^{-\frac{1}{2}}(\Gamma_N)$ find $u \in H^1(D)$ satisfying the equations of linear elasticity

$$-\mathrm{div}\;\sigma(u) = 0 \;\text{ in }\; D, \tag{2.2}$$

$$u = 0 \;\text{ on }\; \Gamma_D, \tag{2.3}$$

$$\sigma(u)n = f \;\text{ on }\; \Gamma_N. \tag{2.4}$$

We assume that $f \neq 0$. The problem has a variational formulation. We define space of displacements

$$V(D) = \{\phi \in H^1(D) : \phi = 0 \text{ on } \Gamma_D\},$$

endowed with the energetic norm

$$\|\phi\|^2_{V(D)} = \int_D \sigma(\phi) : \varepsilon(\phi)\,dx.$$

Indeed, since for elements of the space $V(D)$ the field of displacements vanishes on $\Gamma_D$, then Korn's inequality and the ellipticity condition (2.1) imply that

$$c^{-1}\|\phi\|^2_{H^1(D)} \le \int_D \sigma(\phi) : \varepsilon(\phi)\,dx \le c\|\phi\|^2_{H^1(D)}. \tag{2.5}$$

Thus, $V(D)$ is isomorphic to a subspace of the Hilbert space $H^1(D)$ The total energy of the body has the form

$$\Pi(\phi) = \frac{1}{2}\int_D \sigma(\phi) : \varepsilon(\phi)\,dx - \int_{\Gamma_N} f\cdot\phi\,ds = \frac{1}{2}\|\phi\|^2_{V(D)} - \int_{\Gamma_N} f\cdot\phi\,ds. \tag{2.6}$$

Then the boundary value problem (2.2) - (2.4) can be formulated as a minimization problem: find $u \in V(D)$ such that the functional $\Pi$ reaches its minimum :

$$\Pi(u) \longrightarrow \inf.$$

Since, according to (2.6), the functional $\Pi(D;\cdot)$ is strictly convex and coercive in the Hilbert space $V(D)$, it implies that there is unique solution $u \in V(D)$ of equilibrium problem with some fixed domain $D$, i.e,

$$\Pi(u) = \inf_{\phi \in V(D)} \Pi(\phi).$$

Also the solution satisfies the following variationnal equation

$$\int_D \sigma(u) : \varepsilon(\phi) = \int_{\Gamma_N} f \cdot \phi \quad \forall \phi \in V(D).$$

**Formulation of the inverse problem** We assume that we can measure displacements on the boundary $\Gamma_N$ and that we have some knowledge on the errors which are caused by the measurement of $g$. Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a complete probability space and assume that $g : \Gamma_N \times \Omega \to \mathbb{R}$ is a random field which belongs to the Bochner space $L^2_{\mathbb{P}}(\Omega, H^{\frac{1}{2}}(\Gamma_N))$

The problem is formulated as follows *find the inclusion $S$ with regular boundary $\Sigma$, and dist$(\Sigma, \Gamma) > 0$, such that there exists a function $u \in H^1(D)$ satisfying* (2.2), (2.3) and (2.4) with the additional condition

$$u = g(\omega) \quad \text{on} \quad \Gamma_N. \tag{2.7}$$

The system (2.2) - (2.7) is an overdetermined boundary value problem since two different boundary conditions are prescribed on the boundary $\Gamma_N$. Notice that the data $(f, g(.))$ may be incompatible for some realizations $\omega$ since we model the measurement error with the process $g$.

**Reformulation of the deterministic problem in term of shape optimization.** Let us consider the deterministic case with fixed $\omega$, then $g(x;\omega) = g(x) \in H^{\frac{1}{2}}(\Gamma_N)$. We now reformulate the inverse problem as a shape optimisation problem. First following Feireisl [8], we define the class of admissible domains for some $h > 0$

$$\mathcal{P}_h = \{x \in \mathbb{R}^2 \bigm| \text{dist}(x, \Sigma) < h\}.$$

Then we introduce class of admissible inclusions $S \subset D$ for some positive $\rho$ and $\eta \in C[0, \infty)$

$$\mathcal{A}_\rho(\eta) = \{S \in D \bigm| |\mathcal{P}_h| \le \eta(h) \text{ for any } h > 0 \text{ and } \text{dist}(\Sigma, \Gamma) > \rho\}, \tag{2.8}$$

where $|\cdot|$ is Lebesgue measure in $\mathbb{R}^2$. The function $\eta(h)$ give us some conditions for boundary of inclusion. For example, taking $\eta(h) = Ch$, where $C$ is some positive constant, we obtain the class of inclusions with a "uniformly bounded perimeter".

Next we consider the auxiliary functions $v$ and $w$, satisfying

$$\begin{aligned} -\text{div}\ \sigma(v) &= 0 & \qquad -\text{div}\ \sigma(w) &= 0 & \qquad &\text{in } D, \\ v &= 0 & w &= 0 & &\text{on } \Gamma_D, \end{aligned} \tag{2.9}$$

$$\sigma(v)n = f \qquad\qquad w = g \qquad\qquad \text{on } \Gamma_N,$$

Then we define Kohn-Vogelius functional $J : \mathcal{A}_\rho(\eta) \to \mathbb{R}$

$$J(S) = \int_D \sigma(v-w) : \varepsilon(v-w)\, dx = \int_{\Gamma_N} (f - \sigma(w)n)\cdot(v-g)\, ds.$$

The reconstruction problem can be formulated as follows shape optimization problem: find domain $S \in \mathcal{A}_\rho(\eta)$ such that functional $J$ reaches its minimum

$$J(S) = \|v-w\|^2_{V(D)} \longrightarrow \inf .$$

Our first result is that this shape optimization admits solution.

**Theorem 1.** *For any $\eta \in C[0,\infty)$, finite $\rho$ and $\alpha \in [0,1]$ there exists at least one solution $S^*$ to the shape optimization problem under considerations, i.e.,*

$$J(S^*) = \inf_{S\in\mathcal{A}_\rho(\eta)} J(S).$$

First, we do not claim the equivalence between the inverse problem and the previous minimization problem (2). If $S^*$ is a minimizer of $J$ such that $J(S^*) = 0$, then $S^*$ is a solution of the inverse problem but Theorem 1 says nothing about the value at the minimizer. It should also be noted that this problem does not have a unique solution in general.

To prove the existence result Theorem 1, we follow the direct method of the calculus of variations. Let us introduce the Hausdorff topology in which we will work:

$$\operatorname{dist}(S_1,S_2) = \max\big\{ \sup_{x\in D\setminus S_1} \operatorname{dist}(x, D\setminus S_2), \sup_{x\in D\setminus S_2} \operatorname{dist}(x, D\setminus S_1)\big\}.$$

We need to recall several Lemmas dues to Feireisl [8].

**Lemma 1.** *Let $S_n \in \mathcal{A}_\rho(\eta)$ be a sequence of open sets such that $S_n \xrightarrow{H} S$. Then $|\bar{S}_n \setminus S| \to 0$.*

**Lemma 2.** *The class of open sets $\mathcal{A}_\rho(\eta$ is closed with respect to the Hausdorff topology, i.e. if $S_n \in \mathcal{A}_\rho(\eta)$ and $S_n \xrightarrow{H} S$, then $S \in \mathcal{A}_\rho(\eta)$.*

Next, we need to show the strong convergence of solutions to problems corresponding to the sequence of inclusions.

**Propostion 1.** *Let sequence of open sets $S_n \in \mathcal{A}_\rho(\eta)$ converges to $S$ in sense of Hausdorff topology, $v_n$ and $w_n$ satisfy conditions* (2.9) *with inclusion $S_n$, then $v_n \to v$ and $w_n \to w$ strongly in $H^1(D)$, where $v$ and $w$ satisfy conditions* (2.9) *for inclusion $S$.*

*Proof.* Let us consider sequence of open sets $S_n$ and corresponding fields $v_n$ and $w_n$ defined by (2.9).

First we need to note that tensor $\mathbb{C}$ can be represented as follows $\mathbb{C} = \mathbb{1}_{D\setminus S}\mathbb{C}_1 + \mathbb{1}_S\mathbb{C}_2$, where $\mathbb{1}$ is corresponding indicator function. From Lemma 1, we deduce that if $S_n \xrightarrow{H} S$ then

$$\mathbb{1}_{D\setminus S_n} \to \mathbb{1}_{D\setminus S} \text{ and } \mathbb{1}_{S_n} \to \mathbb{1}_S \text{ in } L^1(D). \tag{2.10}$$

Notice then that there exists some function $G \in V(D)$ such that $G|_{\Gamma_N} = g$ and $\bar{w}_n = w_n - G$ belongs to $H_0^1(D)$.

We can formulate relations (2.5) both for $\bar{w}_n$ and $v_n$ from which we conclude that there exist $v \in H^1(D)$ and $w \in H_0^1(D)$ such that $\varepsilon(v_n) \rightharpoonup \varepsilon(v)$ and $\varepsilon(\bar{w}_n) \rightharpoonup \varepsilon(w)$. Then, by (2.10), we can obtain $\mathbb{C}_n\varepsilon(\phi) \to \mathbb{C}\varepsilon(\phi)$ in $L^2(D)$ for all $\phi \in H^1(D)$, where $\mathbb{C}_n = \mathbb{1}_{D\setminus S_n}\mathbb{C}_1 + \mathbb{1}_{S_n}\mathbb{C}_2$.

Considering all of the above, we can write the following

$$\lim_{n\to\infty}\int_D \mathbb{C}_n\varepsilon(v_n) : \varepsilon(v_n)\,dx = \lim_{n\to\infty}\int_{\Gamma_N} f\cdot v_n\,ds = \int_{\Gamma_N} f\cdot v\,ds = \int_D \mathbb{C}\varepsilon(v) : \varepsilon(v)\,dx$$

and

$$\lim_{n\to\infty}\int_D \mathbb{C}_n\varepsilon(\bar{w}_n) : \varepsilon(w_n)\,dx = \int_D \mathbb{C}\varepsilon(w) : \varepsilon(w)\,dx,$$

It allows us to conclude strong converges of $v_n$ and $w_n$ to $v$ and $w$ which satisfy conditions (2.9) for inclusion $S$ in weak sense, what finalize the proof. □

Finally we can prove easily Theorem 1. From Proposition 1 we obtain continuity of $J$ and from Lemma 2 we get compactness of $\mathcal{A}_\rho(\eta)$, after what we can conclude to the existence of a solution to the minimization problem.

**Random model** Since the data $g \in L^2_{\mathbb{P}}(\Omega, H^{\frac{1}{2}}(\Gamma_N))$ is random $w(\omega)$ also will be a random field from $L^2_{\mathbb{P}}(\Omega, H^1(D))$. It satisfies by linearity of the considering equations. Consequently, the functional $J(S;\omega)$ becomes a random process.

$$J(S;\omega) = \int_D \sigma(v - w(\omega)) : \varepsilon(v - w(\omega))\,dx = \int_{\Gamma_N} (f - \sigma(w(\omega))n)\cdot(v - g(\omega))\,ds.$$

To tackle the problem considering the noise, we choose to minimize a combination of the expectation and the variance of the shape functional $J$. More precisely, we seek an inclusion $S$ (inside the domain $D$) in *argmin* $F_\alpha$ where $F_\alpha$ defined as

$$F_\alpha(S) = (1-\alpha)\mathbb{E}[J(S;\omega)] + \alpha\mathbb{V}[J(S;\omega)]$$

with $\alpha \in [0, 1]$.

Next we make the assumption that the Dirichlet data $g$ corresponds to a centered stochastic process, that can be represented as $\sum_{i=1}^{\infty} g_i(x)Y_i(\omega)$, where the random variables $Y_i(\omega)$ are independent and identically distributed random variables, $Y_i \sim Y$, being centred, $\mathbb{E}[Y] = 0$, normalized, $\mathbb{V}[Y] = 1$, and having finite fourth order moments. In order to simplify the theoretical arguments to get existence and the numerical resolution, we assume in the present work that $g$ can be presented with a finite dimensional noise in the form

$$g(x;\omega) = g_0(x) + \sum_{i=1}^{M} g_i(x) Y_i(\omega),$$

where $M > 0$ some integer. From this expression, we immediately get identities

$$\mathbb{E}[g(x;\omega)] = g_0(x) \quad \text{and} \quad \mathbb{V}[g(x;\omega)] = \sum_{i=1}^{M} g_i^2(x). \tag{2.11}$$

The linearity of equations implies

$$w(x;\omega) = w_0(x) + \sum_{i=1}^{M} w_i(x) Y_i(\omega), \tag{2.12}$$

where $w_i,\ \ i = 0..M$ solves

$$\begin{aligned} -\text{div}\ \sigma(w_i) &= 0 \quad \text{in} \quad D, \\ w_i &= 0 \quad \text{on} \quad \Gamma_D, \\ w_i &= g_i \quad \text{on} \quad \Gamma_N. \end{aligned} \tag{2.13}$$

Then, we can compute expectation and the variance of functional the random shape $J(S;\omega)$ from deterministic quantities.

**Propostion 2** (Expression of the expectation)**.** *It holds*

$$\mathbb{E}[J(S;\omega)] = \int_{\Gamma_N} \big( (f - \sigma(w_0)n) \cdot (v - g_0) + \sum_{i=1}^{M} \sigma(w_i)n \cdot g_i \big) ds.$$

**Propostion 3** (Expression of the variance)**.** *It holds*

$$\begin{aligned} \mathbb{V}[J(S;\omega)] = &(\mathbb{E}[Y^4] - 1) \sum_{i=1}^{M} \left( \int_{\Gamma_N} \sigma(w_i)n \cdot g_i \, ds \right)^2 \\ &- 4\mathbb{E}[Y^3] \sum_{i=1}^{M} \left( \int_{\Gamma_N} \sigma(w_i)n \cdot g_i \, ds \right) \left( \int_{\Gamma_N} \sigma(w_i)n \cdot (v - g_0) \, ds \right) \\ &+ 2 \sum_{i,j=1}^{M} \left( \int_{\Gamma_N} \sigma(w_i)n \cdot g_j \, ds \right)^2 + 4 \sum_{i=1}^{M} \left( \int_{\Gamma_N} \sigma(w_i)n \cdot (v - g_0) \, ds \right)^2 . \end{aligned}$$

The proofs of theses two Propositions are postponed to Section 6.

**Corollary 1.** *If $g$ is a Gaussian random field, then*

$$\mathbb{V}[J(S;\omega)] = 2 \sum_{i,j=1}^{M} \left( \int_{\Gamma_N} \sigma(w_i)n \cdot g_j \, ds \right)^2 + 4 \sum_{i=1}^{M} \left( \int_{\Gamma_N} \sigma(w_i)n \cdot (v - g_0) \, ds \right)^2$$

*Proof.* It obviously follows from proposition (3) and the fact that if $Y \sim \mathcal{N}(0,1)$ then $\mathbb{E}[Y^4] = 3$ and $\mathbb{E}[Y^3] = 0$. $\square$

Finally we can fully formulate the robust obstacle reconstruction problem:

*for $f \in H^{-\frac{1}{2}}(\Gamma_N)$ and $g_i \in H^{\frac{1}{2}}(\Gamma_N)$, $i = 0..M$ find the inclusion $S \in \mathcal{A}_\rho(\eta)$, defined by (2.8), such that $F_\alpha(S) \longrightarrow \inf$, where $F_\alpha$ is the convex combination of the first moments:*

$$F_\alpha(S) = (1-\alpha)\mathbb{E}[J(S;\omega)] + \alpha\mathbb{V}[J(S;\omega)]$$

In terms of the data, it can be written as

$$F_\alpha(S) = (1-\alpha)\int_{\Gamma_N} \big((f - \sigma(w_0)n)\cdot(v - g_0) + \sum_{i=1}^{M} \sigma(w_i)n \cdot g_i\big)ds + \tag{2.14}$$

$$\alpha\left(2\sum_{i,j=1}^{M}\left(\int_{\Gamma_N} \sigma(w_i)n \cdot g_j\, ds\right)^2 + 4\sum_{i=1}^{M}\left(\int_{\Gamma_N} \sigma(w_i)n\cdot(v - g_0)\, ds\right)^2\right), \tag{2.15}$$

$v$ and $w_i$, $i = 0..M$ satisfy (2.9) and (2.13), respectively. Similarly to the exitence Theorem 1 for the deterministic case, we also have the following statement following exitence result for the random averaged case.

**Theorem 2.** *For any $\eta \in C[0,\infty)$, finite $\rho$ and any $\alpha \in [0,1]$ there exists at least one solution $S^*$ to the robust obstacle reconstruction problem under considerations provided that $g$ is a Gaussian random field, i.e.,*

$$F_\alpha(S^*) = \inf_{S \in \mathcal{A}_\rho(\eta)} F_\alpha(S) = \inf_{S \in \mathcal{A}_\rho(\eta)} \mathbb{E}[J(S;\omega)].$$

*Proof.* Let us consider functional

$$\Phi(S) = \int_D \sigma(w_i) : \varepsilon(v),$$

where $v$ and $w_i$ defined by (2.9) and (2.13). From Proposition 1 we can conclude that this $\Phi$ is continuous, which give us continuously of $\mathbb{E}[J(S;\omega)]$. Then it is obviously that $\Phi^2$ is continuous too, which give us continuously of $\mathbb{V}[J(S;\omega)]$ in case of Gaussian random field $g$. Since $F_\alpha$ is convex combination of expectation and variance of $J$ we conclude Theorem statement. $\square$

# 3 Shape calculus

In this section, we will compute the shape gradient of the shape functional to get the necessary conditions of the optimal inclusion. Let us consider transformation of area $D$: $\Psi_t$, which depends on the parameter, such that $\Psi_0(D) = D$. Then, the following limit:

$$dJ(S)\langle\theta\rangle = \lim_{t\to 0} \frac{J(S_t) - J(S)}{t},$$

where $S_t = \Psi_t(S)$ is called by the *shape gradient* of $J$. For more details, we refer to the monographs [5, 10].

### 3.1 Basic identities

Let $\theta : D \to \mathbb{R}^2$ be a $C^\infty$ vector field such that $\theta$ vanishes in a neighborhood of $\Gamma$, i.e., $\theta \in C_0^\infty(D)$. There is $t_0 > 0$, depending on $\theta$ such that for every $t \in [0, t_0]$ the mapping

$$\Psi_t(x) = x + t\theta(x), \quad x \in D,$$

takes diffeomorphically the domain $D$ onto itself. Moreover,

$$\nabla\Psi_t = I + t\nabla\theta$$

admits the estimates

$$0 < c^{-1} \le |(\nabla\Psi_t)^{-1}| \le c, \quad t \in [0, t_0],$$

where $c$ is independent of $t$. We set

$$u = u_0, \quad U_t = u_t \circ \Psi_t \text{ and } \dot{U} = \frac{\partial}{\partial t} U_t\Big|_{t=0} \tag{3.1}$$

Obviously $U_0 = u$. The quantity $\dot{U}$ is the classical *material derivative* (see more details in [17]).

Next define the family of inclusions $S_t$, $t \in [0, t_0]$ by the equality $S_t = \Psi_t(S)$. The corresponding tensor $C_t$ and the stress tensor $\sigma_t(u)$ are defined by the equalities

$$\mathbb{C}_t = \mathbb{1}_{D\setminus S_t}\,\mathbb{C}_1 + \mathbb{1}_{S_t}\,\mathbb{C}_2 \text{ and } \sigma_t(u) = C_t\varepsilon(u).$$

**Lemma 3.** *Under the above assumptions, we have*

$$\begin{aligned} &\frac{d}{dt}\int_D \sigma_t(u_t(y)) : \varepsilon(h_t(y))\,dy\Big|_{t=0} = \\ &\int_D \big(\sigma(U') - \mathbb{C}E(u,\theta) + \text{ div } \theta\,\sigma(u)\big) : \varepsilon(h)\,dx + \int_D \sigma(u) : \big(\varepsilon(H') - E(h,\theta)\big)\,dx, \end{aligned} \tag{3.2}$$

*where*

$$E(u,\theta) = \frac{1}{2}\big(\nabla u\,\nabla\theta + (\nabla u\,\nabla\theta)^\top\big), \quad u = u_0, \quad \mathbb{C} = \mathbb{C}_0 \text{ and } \sigma = \sigma_0.$$

Its proof is postponed to Section 6. In order to derive the equations for shape derivatives of solutions to the elasticity equations we will use the following identity.

**Lemma 4.** *Let all assumptions of Lemma 3 be satisfied.Let $\varphi \in V(D)$ and*

$$h_t(y) = \varphi \circ \Psi_t^{-1}(y).$$

*Then*

$$\begin{aligned} &\frac{d}{dt}\int_D \sigma_t(u_t(y)) : \varepsilon(h_t(y))\,dy\Big|_{t=0} = \\ &\qquad\int_D \big(\sigma(U') - \mathbb{C}E(u,\theta) + \text{ div } \theta\,\sigma(u)\big) : \varepsilon(\varphi)\,dx \int_D \mathbf{b}(u,\theta) : \nabla\varphi\,dx, \end{aligned} \tag{3.3}$$

*where*

$$\mathbf{b}(u,\theta) = \sigma(u)\,(\nabla\theta)^\top.$$

*Proof.* We begin with the observation that $H_t = \varphi(x)$ is independent of $t$. It follows from this $H' = 0$ and $h = \varphi$. Hence identity (3.2) reads

$$\frac{d}{dt}\int_D \sigma_t(u_t(y)) : \varepsilon(h_t(y))\,dy\Big|_{t=0} =$$

$$\int_D \big(\sigma(U') - \mathbb{C}E(u,\theta) + \text{ div }\theta\,\sigma(u)\big) : \varepsilon(\varphi)\,dx \;-\int_D \sigma(u) : E(\varphi,\theta)\,dx, \tag{3.4}$$

Next notice that

$$\begin{aligned}\sigma(u) : E &= \frac{1}{2}\sigma_{ij}\left(\nabla\varphi_{ik}\nabla\theta_{kj} + \nabla\varphi_{jk}\nabla\theta_{ki}\right) = \sigma_{ij}\nabla\varphi_{ik}\nabla\theta_{kj}\\ &= (\sigma\nabla\theta^\top)_{ik}\nabla\varphi_{ik} = (\sigma\nabla\theta^\top) : \nabla\varphi = \mathbf{b}(u,\theta) : \nabla\varphi.\end{aligned}$$

Substituting this equality in (3.4) we obtain desired identity (3.3).

□

Next we derive formulas for material derivatives of solutions to two basic boundary value problems.

**Lemma 5.** *For a given vector field $f \in H^{-\frac{1}{2}}(\Gamma_N)$ consider a mixed boundary value problem for linear elasticity equations in variational formulation*

$$\int_D \sigma_t(v_t) : \varepsilon(\varphi)\,dx = \int_{\Gamma_N} f\cdot\varphi\,ds \quad \forall\phi \in V(D). \tag{3.5}$$

*Then material derivative $V'$ satisfies*

$$\int_D \big(\sigma(V') - \mathbb{C}E(v,\theta) + \text{ div }\theta\,\sigma(v)\big) : \varepsilon(\varphi)\,dx \;-\int_D \mathbf{b}(v,\theta) : \nabla\varphi\,dx = 0 \quad \forall\phi \in V(D). \tag{3.6}$$

*Proof.* Differentiating both sides of equality (3.5) with respect to $t$ at $t = 0$ and applying Lemma 4 we arrive at the necessary equality (3.6). □

**Lemma 6.** *For a given vector field $g \in H^{\frac{1}{2}}(\Gamma)$ such that $g = 0$ on $\Gamma_D$, consider Dirichlet boundary value problem for linear elasticity equations in variational formulation.*

$$\int_D \sigma_t(v_t) : \varepsilon(\varphi)\,dx = 0 \quad \forall\phi \in V_g(D), \tag{3.7}$$

*where*

$$V_g(D) = \{\phi \in H^1(D) : \phi = g \text{ on } \Gamma\}.$$

*Then the material derivative $\dot{W}$ satisfies*

$$\begin{aligned}\int_D \sigma(\dot{W}) : \epsilon(\varphi)\,dx &= \int_D (\mathbb{C}E(w,\theta) - div\ \theta\sigma(w)) : \varepsilon(\varphi)\,dx\\ &+ \int_D \mathbf{b}(w,\theta) : \nabla\varphi\,dx + \int_{\Gamma_N} (\sigma(\dot{W})n)\cdot\varphi\,ds \quad \forall\phi \in V(D).\end{aligned} \tag{3.8}$$

*Proof.* Differentiating both sides of equality (3.7) with respect to $t$ at $t = 0$ and applying Lemma 4 we arrive at the equality

$$\int_D \big( \sigma(W') - \mathbb{C}E(w, \theta) + \text{ div } \theta \, \sigma(w) \big) : \varepsilon(\varphi) \, dx - \int_D \mathbf{b}(w, \theta) : \nabla\varphi \, dx = 0 \quad \forall \phi \in V_g(D). \tag{3.9}$$

Integral identity (3.9) holds for all functions $\varphi \in V_g(D)$. However, our main functional space is $V(D)$. Let us now consider the extension of (3.9) to $\varphi \in V(D)$. Set

$$\mathbf{A} = \mathbb{C}E(w, \theta) - \text{div } \theta \, \sigma(w) + \mathbf{b}(w, \theta).$$

Then, by Green's formula, we obtain

$$\int_D \sigma(W') : \epsilon(\varphi) \, dx = - \int_D \text{ div } \mathbf{A} \cdot \varphi \, dx + \int_\Gamma (\sigma(W')n) \cdot \varphi \, ds. \tag{3.10}$$

Since $\mathbf{A}$ vanishes in a neighbourhood of $\Gamma$, we have

$$\begin{aligned} - \int_D \text{ div } \mathbf{A} \cdot (\varphi) \, dx &= \int_D \mathbf{A} : \nabla(\varphi) \, dx \\ &= \int_D (\mathbb{C}E(w, \theta) - \text{div } \theta \, \sigma(w)) : \nabla(\varphi) \, dx + \int_D \mathbf{b}(w, \theta) : \nabla(\varphi) \, dx \\ &= \int_D (\mathbb{C}E(w, \theta) - \text{div } \theta \, \sigma(w)) : \varepsilon(\varphi) \, dx + \int_D \mathbf{b}(w, \theta) : \nabla(\varphi) \, dx. \end{aligned} \tag{3.11}$$

Substituting this equality into (3.10), we finally arrive at the identity

$$\begin{aligned} \int_D \sigma(W') : \epsilon(\varphi) \, dx = &\int_D (\mathbb{C}E(w, \theta) - \text{div } \theta \, \sigma(w)) : \varepsilon(\varphi) \, dx \\ &+ \int_D \mathbf{b}(w, \theta) : \nabla\varphi \, dx + \int_{\Gamma_N} (\sigma(W')n) \cdot \varphi \, ds, \end{aligned}$$

which holds true for all $\varphi \in V(D)$. □

## 3.2 Shape gradient

After proving all the necessary assumptions finally we can calculate shape gradient of functional $J(S)$, i.e. in deterministic case with fixed event.

**Theorem 3** (Volume expression of shape gradient)**.** *The shape gradient of $J(S)$ has following expression*

$$\begin{aligned} dJ(S)\langle\theta\rangle = 2 &\int_D \sigma(v) : E(v, \theta) \, dx - 2 \int_D \sigma(w) : E(w, \theta) \, dx \\ &- \int_D \text{div } \theta \big( \sigma(v) : \varepsilon(v) - \sigma(w) : \varepsilon(w) \big) \, dx. \end{aligned} \tag{3.12}$$

The last step in the derivation of the formula for $dJ$ is to obtain the canonical boundary expression of the shape gradient according to the structure Theorem [10].

**Corollary 2** (Boundary expression of the shape gradient)**.** *It holds*

$$dJ(S)\langle\theta\rangle = 2\int_{\Sigma} \big((\sigma(v)n)\cdot[\partial_n(v)] - (\sigma(w)n)\cdot[\partial_n(w)]\big)\theta\cdot n\,ds \\ - \int_{\Sigma}[\sigma(v):\varepsilon(v) - \sigma(w):\varepsilon(w)]\,\theta\cdot n\,ds.$$

Here we can see the main feature of the Kohn-Vogelius functional: the absence of derivatives of the solutions of the state equation in the expression of the shape gradient. This fact greatly facilitates the numerical implementation. These results can be compared with those with those of [1].

Now we can write necessary condition of an optimal solution $S^*$ of shape optimization problem with deterministic additional data $g(x) \in H^{\frac{1}{2}}(\Gamma_N)$ for every sufficiently smooth variation fields $\theta$

$$dJ(S^*)\langle\theta\rangle = 0.$$

To finalize computing of shape gradient we need to compute shape gradient of the expectation.

**Propostion 4.** *It holds*

$$\begin{aligned}\frac{d}{dt}(\mathbb{E}[J(S;\omega)])\langle\theta\rangle =& 2\int_D \sigma(v):E(v,\theta)\,dx - 2\int_D\sum_{i=0}^{M}\sigma(w_i):E(w_i,\theta)\,dx \\ & - \int_D \operatorname{div}\theta\,\Big(\sigma(v):\varepsilon(v) - \sum_{i=0}^{M}\sigma(w_i):\varepsilon(w_i)\Big)\,dx \\ =& \int_{\Sigma}[\sigma(v):\varepsilon(v) - \sum_{i=0}^{M}\sigma(w_i):\varepsilon(w_i)]\,\theta\cdot n\,ds \\ & - 2\int_{\Sigma}\big((\sigma(v)n)\cdot[\partial_n(v)] - \sum_{i=0}^{M}(\sigma(w_i)n)\cdot[\partial_n(w_i)]\big)\theta\cdot n\,ds.\end{aligned}$$

*Proof.* Proof is based on the fact that

$$\frac{d}{dt}(\mathbb{E}[J(S;\omega)])\langle\theta\rangle = \mathbb{E}\left[\frac{d}{dt}J(S;\omega)\langle\theta\rangle\right].$$

We then repeat technique from proof of Proposition 2 to result of Theorem 3 and Corollary 2 which give us required expression. □

The derivative of variance looks a little more complicated.

**Propostion 5** (Shape gradient of the variance)**.** *It holds*

$$
\begin{aligned}
&\frac{d}{dt}(\mathbb{V}[J(S;\omega)])\langle\theta\rangle = \\
&\quad 4\sum_{i,j=1}^{M}\Big(\int_{\Gamma_N}\sigma(w_i)n\cdot g_j\,ds\Big)\Big(2\int_D\sigma(w_i):E(w_j,\theta)\,dx\, -\int_D \operatorname{div}\,\theta\,\sigma(w_i):\varepsilon(w_j)\,dx\Big)\\
&\quad +8\sum_{i=1}^{M}\Big(\int_{\Gamma_N}\sigma(w_i)n\cdot(v-g_0)\,ds\Big)\Big(2\int_D\sigma(w_i):E(v,\theta)\,dx-2\int_D\sigma(w_i):E(w_0,\theta)\,dx\\
&\qquad\qquad -\int_D\operatorname{div}\,\theta\,\big(\,\sigma(w_i):\varepsilon(v)-\sigma(w_i):\varepsilon(w_0)\,\big)\,dx\Big)\\
&\quad =4\sum_{i,j=1}^{M}\Big(\int_{\Gamma_N}\sigma(w_i)n\cdot g_j\,ds\Big)\Big(\int_\Sigma[\sigma(w_i):\varepsilon(w_j)]\,\theta\cdot n\,ds-2\int_\Sigma(\sigma(w_i)n)\cdot[\partial_n(w_j)]\theta\cdot n\,ds\Big)\\
&\quad +8\sum_{i=1}^{M}\Big(\int_{\Gamma_N}\sigma(w_i)n\cdot(v-g_0)\,ds\Big)\Big(\int_\Sigma[\sigma(w_i):\varepsilon(v)-\sigma(w_i):\varepsilon(w_0)]\,\theta\cdot n\,ds\\
&\qquad\qquad -2\int_\Sigma\big((\sigma(w_i)n)\cdot[\partial_n(v)]-(\sigma(w_i)n)\cdot[\partial_n(w_0)]\big)\theta\cdot n\,ds\Big).
\end{aligned}
$$

Finally we can write shape gradient of $F_\alpha(S)$ in area integral form in the case of $g$ is Gaussian random field in boundary integral form

$$
\begin{aligned}
dF_\alpha(S)\langle\theta\rangle &= (1-\alpha)d(\mathbb{E}[J(S;\omega)])\langle\theta\rangle+\alpha d(\mathbb{V}[J(S;\omega)])\langle\theta\rangle\\
&=(1-\alpha)\Big(2\int_\Sigma\big((\sigma(v)n)\cdot[\partial_n(v)]-\sum_{i=0}^{M}(\sigma(w_i)n)\cdot[\partial_n(w_i)]\big)\theta\cdot n\,ds\\
&-\int_\Sigma[\sigma(v):\varepsilon(v)-\sum_{i=0}^{M}\sigma(w_i):\varepsilon(w_i)]\,\theta\cdot n\,ds\Big)\\
&+\alpha\Big(4\sum_{i,j=1}^{M}\Big(\int_{\Gamma_N}\sigma(w_i)n\cdot g_j\,ds\Big)\Big(\int_\Sigma[\sigma(w_i):\varepsilon(w_j)]\,\theta\cdot n\,ds\,-2\int_\Sigma(\sigma(w_i)n)\cdot[\partial_n(w_j)]\theta\cdot n\,ds\Big)\\
&+8\sum_{i=1}^{M}\Big(\int_{\Gamma_N}\sigma(w_i)n\cdot(v-g_0)\,ds\Big)\Big(\int_\Sigma[\sigma(w_i):\varepsilon(v)-\sigma(w_i):\varepsilon(w_0)]\,\theta\cdot n\,ds\\
&\qquad\qquad -2\int_\Sigma\big((\sigma(w_i)n)\cdot[\partial_n(v)]-(\sigma(w_i)n)\cdot[\partial_n(w_0)]\big)\theta\cdot n\,ds\Big)\Big).
\end{aligned}
$$

# 4 Numerical examples.

In this part, we will give a numerical method for solving the problem and a comparison of various optimization methods, thanks to which we can better see the various features of the problem under consideration.

### 4.1 Discretization

Since the considering problem is infinite dimensional optimization problem we are not able to solve it directly. Thus we replace it by a finite dimensional problem. For the numerical computations, we restrict ourselves to inclusions which are star-shaped with respect to the origin 0. Then, the boundary of an inclusion can be parametrized by the Fourier series based on polar coordinates. Hence, it is reasonable to approximate the radial function by a truncated Fourier series

$$r_N(\phi) = a_0 + \sum_{n=1}^{N} a_n \cos(n\phi) + a_{-n}\sin(n\phi).$$

Since $r_N$ admits $2N+1$ degrees of freedom we can reformulate problem as optimisation problem in set

$$A_N = \{\mathbf{a} = (a_{-N}, a_{1-N}, ..., a_N) \in \mathbb{R}^{2N+1}\}.$$

We can set a correspondence between the vector $\mathbf{a}$ and the inclusion S, then we can write that $J(S) = J(S(\mathbf{a})) = \tilde{J}(\mathbf{a})$. Finally for numerical realization we will consider following problem: find vector $\mathbf{a} \in A_N$ such that functional $\tilde{J}$ reaches its minimum

$$\tilde{J}(\mathbf{a}) \longrightarrow \inf .$$

In what follows, we will use the notation $J$ instead of $\tilde{J}$

Usually for solving optimization problems applied classic gradient descent method

$$\mathbf{a}^{t+1} = \mathbf{a}^t - h^t dJ(\mathbf{a}^t),$$

with convergence rate $O(1/t)$.

But as noted earlier, the problem is extremely ill-posed. In our case we can say that the considered objective functional is practically "flat", which is why the gradient descent method is not efficient enough, which will be shown below. Therefore, we will consider other more effective methods for, or rather methods based on the Nesterov's method

$$\mathbf{a}^{t+1} = \tilde{\mathbf{a}}^t - h^t dJ(\tilde{\mathbf{a}}^t),$$

$$\tilde{\mathbf{a}}^{t+1} = \mathbf{a}^{t+1} + \frac{t}{t+3}(\mathbf{a}^{t+1} - \mathbf{a}^t), \quad \tilde{\mathbf{a}^0} = \mathbf{a}^0.$$

It should be noted that this method is theoretically when funded only in the case of a convex objective functional, which is the considered Kohn-Vogelius functional $J$. Converges rate is in that case the famous $O(1/t^2)$ (see [14]). This method is much faster, but it is not a descent method, i.e. it can be that $J(\mathbf{a}^{t+1}) > J(\mathbf{a}^t)$. Which is a significant difficulty, since on the way of minimizing the functional, intermediate solutions can correspond to an unacceptable form, which will immediately stop the algorithm. To avoid this problem we will use adaptive restart method

$$\mathbf{a}^{t+1} = \tilde{\mathbf{a}}^t - h^t dJ(\tilde{\mathbf{a}}^t),$$
$$\tilde{\mathbf{a}}^{t+1} = \mathbf{a}^{t+1} + \frac{\eta_t - 1}{\eta_{t+1}}(\mathbf{a}^{t+1} - \mathbf{a}^t), \quad \tilde{\mathbf{a}^0} = \mathbf{a}^0,$$
$$\eta_{t+1} = \frac{1 + \sqrt{1 + 4\eta_t^2}}{2}, \quad \eta_0 = 1$$
$$\text{with restart, if } J(\mathbf{a}^t) > J(\mathbf{a}^{t-1}) : \mathbf{a}^0 \leftarrow \mathbf{a}^t, \quad \tilde{\mathbf{a}}^0 \leftarrow \mathbf{a}^t, \quad \eta_0 \leftarrow 1.$$

Coefficient $\frac{\eta_t - 1}{\eta_{t+1}} = 1 - \frac{3}{t} + o\big(\frac{1}{t}\big)$, which is asymptotically equivalent to $\frac{t}{t+3}$, which is present in Nesterov's method. Converges rate is same as for Nesterov's method $O(1/t^2)$. For more details look in [12].

The associated gradient has to be computed with respect to all directions under consideration:

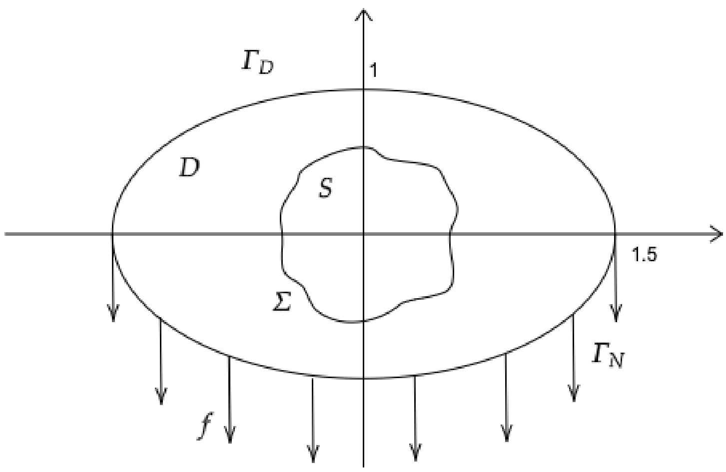
$$\theta(\phi) = \sin(N\phi)\mathbf{e}_r(\phi), \sin((N-1)\phi)\mathbf{e}_r(\phi), .., \cos(N\phi)\mathbf{e}_r(\phi),$$

where $\mathbf{e}_r(\phi) = (\cos\phi, \sin\phi)$.

## 4.2 Numerical experiments

The region was chosen in the form of an ellipse with $r_x = 1.5$ and $r_y = 1$. The inclusion $S$ with unknown shape considered with known center located at the point $(0, 0)$. We put $f = (0, -1)$.

On the lower half of the ellipse, the Neumann condition was imposed, and on the remaining zero Dirichlet condition, see figure below.




Model of the plane-stressed state of the Lame of an isotropic solid is given in terms of the stress tensor:

$$\sigma_{11}(u) = (2\mu+\lambda)\varepsilon_{11}(u) + \lambda\varepsilon_{22}(u), \quad \sigma_{12}(u) = \sigma_{21}(u) = 2\mu\varepsilon_{12}(u),$$

$$\sigma_{22}(u) = \lambda\varepsilon_{11}(u) + (2\mu+\lambda)\varepsilon_{22}(u),$$

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{2\mu\nu}{1-2\nu},$$

Was chosen the following material parameters of the part of the body $D \setminus S$:

$$\nu_1 = 0.2, \quad E_1 = 15,$$

and inclusion $S$:

$$\nu_2 = 0.35, \quad E_2 = 40.$$

We put $N = 3$. We chose small number of coefficients, since with a large number, significant difficulties are observed even with the adaptive restart method. The main problems are self-crossing of boundaries during the operation of the algorithm. Usually, in identification problems in the case of elasticity simpler cases are considered. For example, the identification of the position of an inclusion with a known shape or the radii of an ellipse (see [13] for example). For Nesterov's and adaptive restart method $h_t = 1$, for gradient descent method $h_t = \mathrm{argmin}_h J(\mathbf{a}^{t+1})$. Finally, we used FreeFem to solve the state problem [9].

#### 4.2.1 Case without noise

First, we considered the problem in the deterministic case, that is, with a fixed event $\omega$, i.e. $g$ choosen as solution to problem with an known inclusion, which is an exact solution to the problem of obstacle reconstruction. We show the advantage of the adaptive restart method over the other methods. The picture 1 shows the values of the functional at each step over 30 iterations for the considering methods. The picture emphasizes how the adaptive restart method converges faster than gradient descent. It is worth noting that the descent speed is constantly decreasing and cannot always achieve the accuracy of the adaptive restart algorithm. One can observe that the oscillations of the Nesterov's method are much stronger than in the case of adaptive restart. It is important to emphasize that after the 30th iteration, the coefficients in the case of the Nesterov's method became such that the form allowed self-intersections, after which the algorithm could not continue working, the adaptive restart method did not allow this.

One can see how the path to the desired shape runs through an impossible geometry with self-suppression in the case of Nesterov's method in the picture 2. In the picture 3 you can see good results of the adaptive restart method. You can also compare the work of the algorithms in picture 1.

The observed results show how ill-posed the problem is. Therefore, the effect of noise will be extremely high in problems of this type.

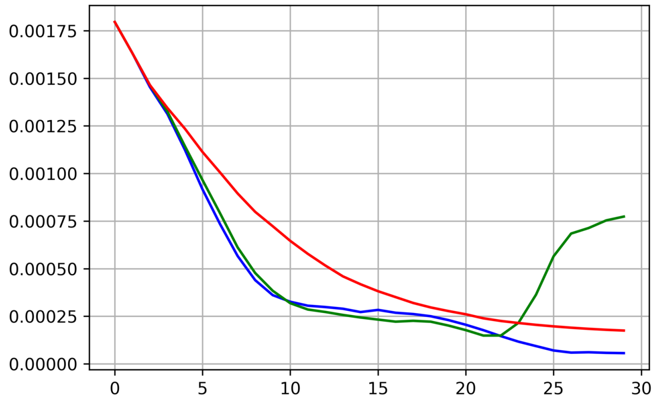



Figure 1: Decrease of the functional $J$. Blue: adaptive restart, green: Nesterov's, red: gradient descent.

#### 4.2.2 Case with noise

For experiments we choose additional noised data as

$$g(x;\omega) = g_e(x) + g_n(x;\omega),$$

where $g_e(x)$ is solution to problem with an inclusion, which is an exact solution to the obstacle reconstruction problem, and $g_n(x,\omega)$ is noise. Noise was modeling by Karhunen-Loeve expansion, see (2.11), where $g_n$ is a Gaussian random field

$$g_n = g_0 + \sum_{i=1}^{M} r_i g_i, \quad g_0 = \frac{1}{M}\sum_{i=1}^{M} g_i \text{ and } g_i = \beta \sin(i\phi), \quad i = 1..M,$$

where $r_i, \quad i = 1..M$ are generated by random numbers and $\beta$ is noise scaling coefficient, which was chosen to get corresponding percent of noise, i.e. $\|g_n\|_L^2(D)$ is corresponding percent of $\|g_e\|_L^2(D)$. We took $M = 10$ in the experiments.

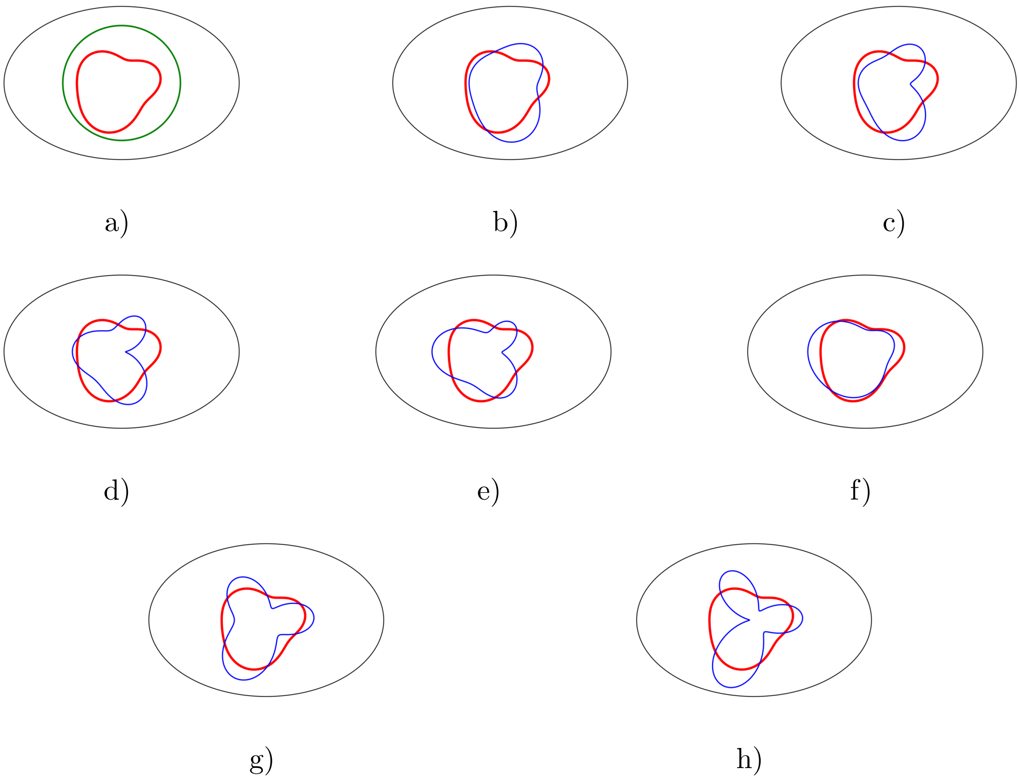



Figure 2: Nesterov's method. Exact - red, initial - green and approximate - blue solutions for different iterations: a) initial position, b) 4 iteration, c) 8 iteration, d) 12 iteration, e) 16 iteration, f) 20 iteration, g) 24 iteration, h) 30 iteration.

In the picture 4, we plot graph of functional $J$ on each step of the adaptive restart method. We observe that classical approach give a great variance of the reconstructions with noise about 5%. Also note that when the noise was more than 5%, even the adaptive restart algorithm stopped converging. The picture 5 shows noise influence on the results after 30 iterations for functional $J$.

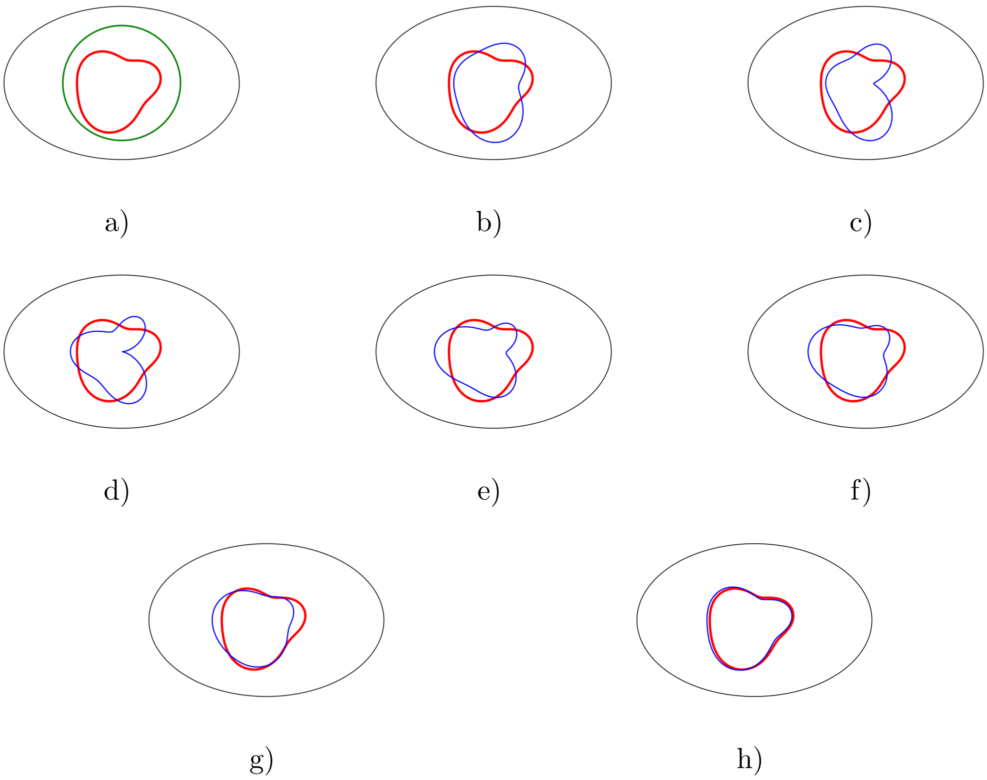



Figure 3: Adaptive restart method. Red: exact, green: initial and blue: approximate solutions for different iterations: a) initial position, b) 4 iteration, c) 8 iteration, d) 12 iteration, e) 16 iteration, f) 20 iteration, g) 24 iteration, h) 30 iteration.

Then, we took $F_\alpha$ as target functional instead of $J$. First we choose $\alpha = 0$, i.e. we work with expectation of Kohn-Vogelius functional. In pictures 6 and 7 you can see how much the situation improves even in the case of 7.5% noise.

**Parameter** $\alpha$ In this paragraph we consider the dependence of the reconstruction algorithm on the parameter $\alpha$. We fixed the noise level 5% and launch algorithm with different $\alpha$: 0, 0.1, 0.2, 0.3, 0.4.

In the picture 8 we observe increasing the parameter improves convergence, but after $\alpha = 0.3$ the algorithm stops converging. The functional becomes more flat as $\alpha$ increases and there is some critical point, after which realization of minimization becomes difficult. Picture 9 shows results after 30 iterations for functional $F_\alpha$.

## 5 Conclusion

In the present work, we have proposed a method that enables to reconstruct inclusions in elastic bodies in both cases: with usual and noised additional data. A comprehensive

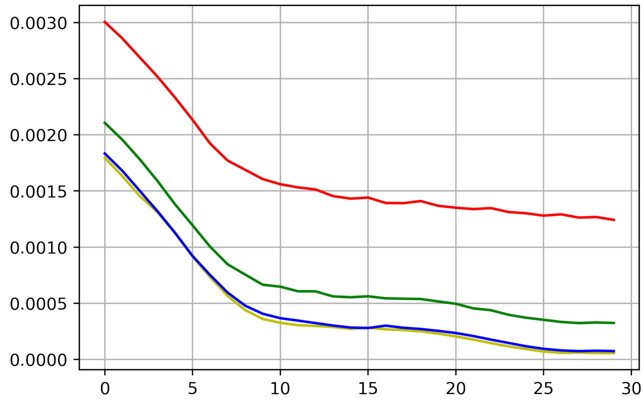
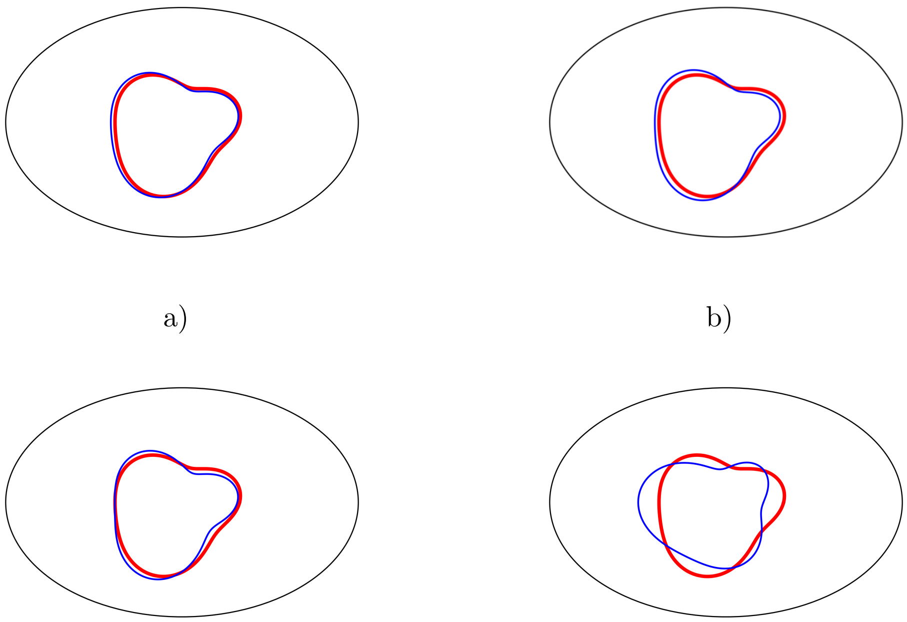



Figure 4: Decrease of the functional $J$ for noise levels. Yellow: 0%, blue: 1%, green: 2.5%, red: 5%.



a) b)

d) e)

Figure 5: Noise influence. Exact - red and approximate - blue solutions for noise levels: a) 0%, b) 1%, c) 2.5%, d) 5%.

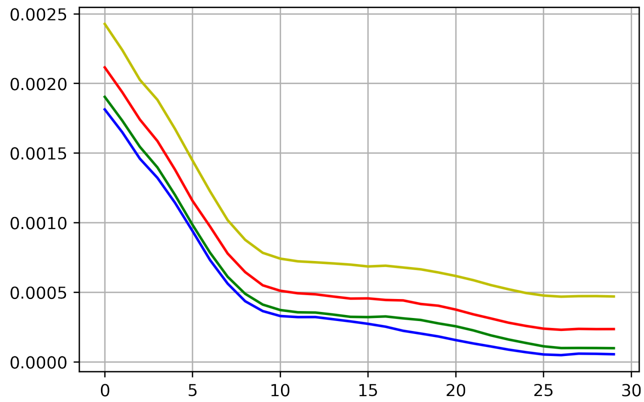
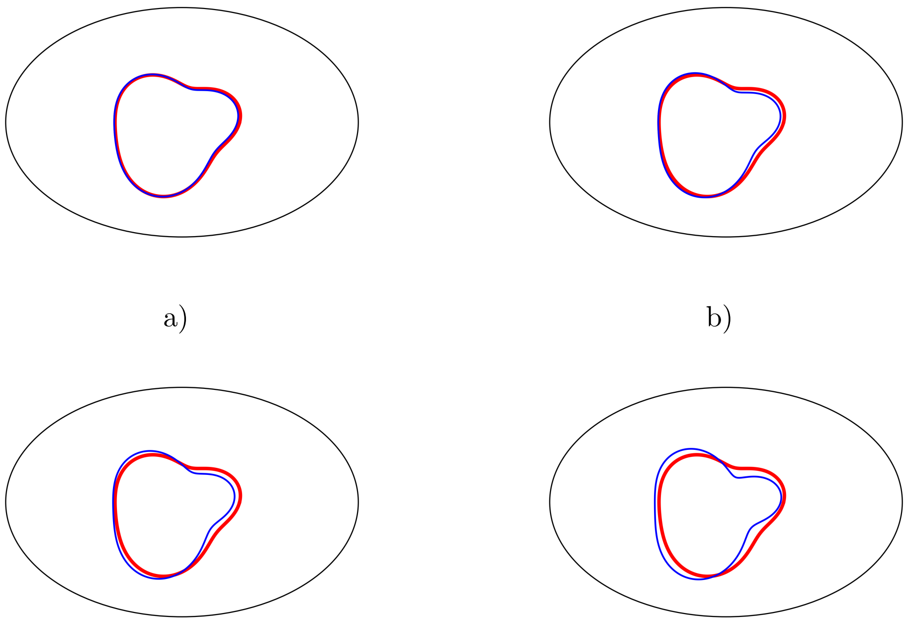



Figure 6: Decrease of the functional $F_0$ for noise levels. Blue: 1%, green: 2.5%, red: 5%, yellow: 7.5%.



a) b) d) e)

Figure 7: Noise influence. Exact - red and approximate - blue solutions for noise levels: a) 1%, b) 2.5%, c) 5%, d) 7.5%.

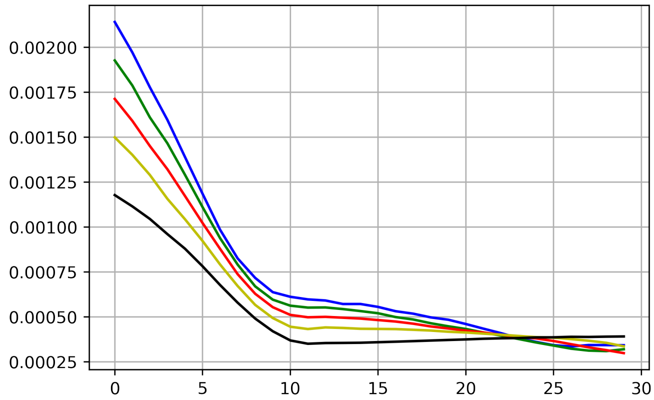



Figure 8: Decrease of the functional $F_\alpha$ for noise level 5% with different $\alpha$. Blue: 0, green: 0.1, red: 0.2, yellow: 0.3, black: 0.4.

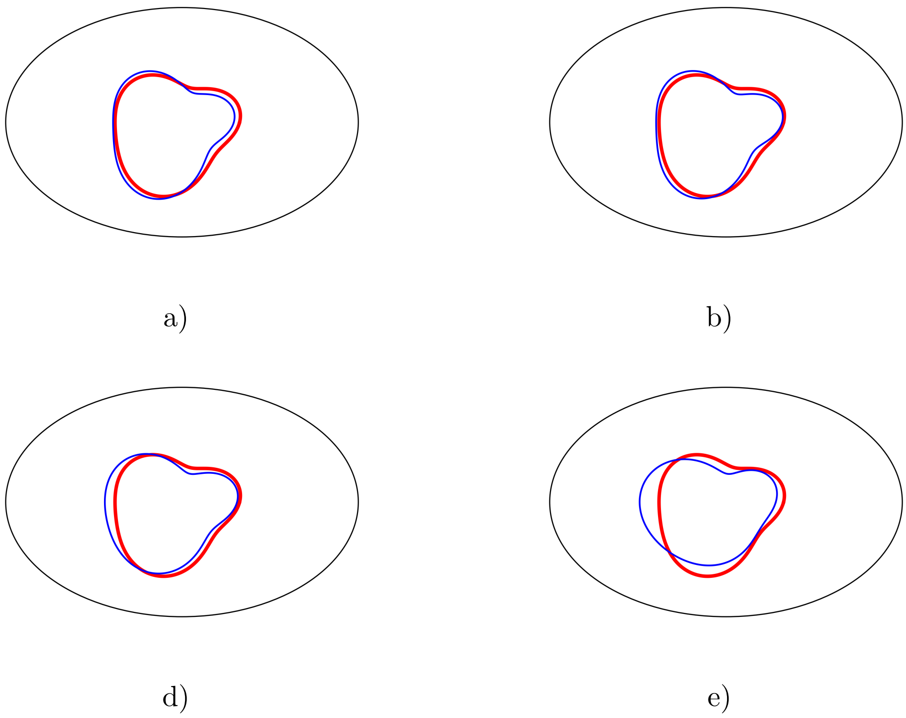



Figure 9: Parameter $\alpha$ influence. Exact - red and approximate - blue solutions for $\alpha$: a) 0, b) 0.1, c) 0.2, d) 0.3.

study of the problem was carried out: the existence of a solution to the problem was presented, a necessary condition for an optimal solution was formulated, a method for numerical implementation was also proposed, and various numerical methods were compared. The incorrectness of the problem was clearly shown. It should be noted that in the problem of identifying inclusions in elasticity models, simple variations of the sought objects are usually considered: an unknown center or ellipse radius. In this paper, more complex shapes of inclusions have been considered, but significant difficulties are still observed. This shows the need for further study of such problems. The main problems in the numerical solution arise from the lack of taking into account the constraints in the algorithm on the coefficients that determine the shape, which can cause self-intersections of boundary. Also, the problem clearly requires a detailed second-order analysis, which was done for simpler models, but not for the elasticity model.

# 6 Technical proofs.

## 6.1 Computation of the expectation of the objective. Proof of Proposition 2.

Since $\sigma(\phi) : \varepsilon(\phi)$ is non negative function from Fubini's theorem we get following expression for the expectation of $J(S;\omega)$

$$
\begin{aligned}
\mathbb{E}[J(S;\omega)] &= \int_\Omega \int_D \sigma(v - w(\omega)) : \varepsilon(v - w(\omega)) \, dx \, d\mathbb{P}(\omega) \\
&= \int_D \int_\Omega \sigma(v - w(\omega)) : \varepsilon(v - w(\omega)) \, d\mathbb{P}(\omega) \, dx \\
&= \int_D \left( \int_\Omega \langle \sigma(v(x) - w(x;\omega)), \varepsilon(v(y) - w(y,\omega)) \rangle \, d\mathbb{P}(\omega) \right)_{x=y} dx.
\end{aligned}
$$

Using form of $w$ (2.12) under assumption that $Y_i,\ i = 1..M$ are independent and identically distributed random variables, we conclude that

$$
\begin{aligned}
&\int_\Omega \langle \sigma(v(x) - w(x;\omega)), \varepsilon(v(y) - w(y,\omega)) \rangle \, d\mathbb{P}(\omega) \\
&= \int_\Omega \langle \sigma(v(x) - w_0(x) - \sum_{i=1}^M w_i(x) Y_i(\omega)), \varepsilon(v(y) - w_0(y) - \sum_{i=1}^M w_i(y) Y_i(\omega)) \rangle \, d\mathbb{P}(\omega) \\
&= \langle \sigma(v(x) - w_0(x)), \varepsilon(v(y) - w_0(y)) \rangle - 2 \sum_{i=1}^M \langle \sigma(w_i(x)), \varepsilon(v(y) - w_0(y)) \rangle \mathbb{E}[Y_i] \\
&\quad + \sum_{i,j=1}^M \langle \sigma(w_i(x)), \varepsilon(w_j(y)) \rangle \mathbb{E}[Y_i Y_j].
\end{aligned}
$$

Then we use $Y_i \sim Y,\ i = 1..M$ which is centred and normalized to arrive at

$$
\mathbb{E}[J(S;\omega)] = \int_D \big( \sigma(v - w_0) : \varepsilon(v - w_0) + \sum_{i=1}^M \sigma(w_i) : \varepsilon(w_i) \big) \, dx.
$$

After using Green's formula we immediately get the expression that was required to prove.

## 6.2 Computation of the variance of the objective. Proof of Proposition 3.

The variance can be computed as difference between uncentered second moment and square of expectation:

$$\mathbb{V}[J(S\ \omega)] = \mathbb{E}[J(S;\omega)^2] - \mathbb{E}^2[J(S;\omega)]. \tag{4.1}$$

The uncentred second moment can be expressed as follows

$$\begin{aligned}\mathbb{E}[J(S;\omega)^2] &= \int_\Omega \left( \int_D \sigma(v - w(\omega)) : \varepsilon(v - w(\omega))\, dx \right)^2 d\mathbb{P}(\omega)\\ &= \int_D \int_D \int_\Omega \big(\sigma(v(x) - w(x;\omega)) : \varepsilon(v(x) - w(x;\omega))\big)\\ &\qquad \big(\sigma(v(y) - w(y;\omega)) : \varepsilon(v(y) - w(y;\omega))\big)\, d\mathbb{P}(\omega)\, dy\, dx.\end{aligned}$$

Using expressions (2.13) and (6.1), we obtain

$$\begin{aligned}&\int_\Omega \big(\sigma(v(x) - w(x;\omega)) : \varepsilon(v(x) - w(x;\omega))\big)\big(\sigma(v(y) - w(y;\omega)) : \varepsilon(v(y) - w(y;\omega))\big)\, d\mathbb{P}(\omega)\\ &\quad = \sum_{i,j,k,l=1}^{M} \langle \sigma(w_i(x)), \varepsilon(w_j(x))\rangle \langle \sigma(w_k(y)), \varepsilon(w_l(y))\rangle \mathbb{E}[Y_i Y_j Y_k Y_l]\\ &\quad - 4 \sum_{i,j,k=1}^{M} \langle \sigma(w_i(x)), \varepsilon(w_j(x))\rangle \langle \sigma(w_k(y)), \varepsilon(v(y) - w_0(y))\rangle \mathbb{E}[Y_i Y_j Y_k]\\ &\quad + \sum_{i,j=1}^{M} \big\{ 2\langle \sigma(w_i(x)), \varepsilon(w_j(x))\rangle \langle \sigma(v(y) - w_0(y))), \varepsilon(v(y) - w_0(y))\rangle\\ &\quad + 4\langle \sigma(w_i(x)), \varepsilon(v(x) - w_0(x))\rangle \langle \sigma(w_j(y)), \varepsilon(v(y) - w_0(y))\rangle \big\} \mathbb{E}[Y_i Y_j]\\ &\quad - 4 \sum_{i=1}^{M} \langle \sigma(w_i(x)), \varepsilon(v(x) - w_0(x))\rangle \langle \sigma(v(y) - w_0(y)), \varepsilon(v(y) - w_0(y))\rangle \mathbb{E}[Y_i]\\ &\quad + \langle \sigma(v(x) - w_0(x)), \varepsilon(v(x) - w_0(x))\rangle \langle \sigma(v(y) - w_0(y)), \varepsilon(v(y) - w_0(y))\rangle\end{aligned}$$

Then from fact that $Y_i \sim Y,\ i = 1..M$ and $\mathbb{E}[Y] = 0,\ \mathbb{V}[Y] = 1$ it holds

$$\begin{aligned}\mathbb{E}[J(S;\omega)^2] = &\,\mathbb{E}[Y^4] \sum_{i=1}^{M} \left( \int_D \sigma(w_i) : \varepsilon(w_i)\, dx \right)^2 + \mathbb{E}[Y^2]^2 \Bigg\{ 2 \sum_{i\neq j}^{M} \left( \int_D \sigma(w_i) : \varepsilon(w_j)\, dx \right)^2\\ &+ \sum_{i\neq j}^{M} \left( \int_D \sigma(w_i) : \varepsilon(w_i)\, dx \right) \left( \int_D \sigma(w_j) : \varepsilon(w_j)\, dx \right) \Bigg\}\end{aligned}$$

$$
\begin{aligned}
&- 4\mathbb{E}[Y^3] \sum_{i=1}^{M} \Big( \int_D \sigma(w_i) : \varepsilon(w_i)\, dx \Big) \Big( \int_D \sigma(w_i) : \varepsilon(v - w_0)\, dx \Big) \\
&+ \mathbb{E}[Y^2] \Big\{ 2 \sum_{i=1}^{M} \Big( \int_D \sigma(w_i) : \varepsilon(w_i)\, dx \Big) \Big( \int_D \sigma(v - w_0) : \varepsilon(v - w_0)\, dx \Big) \\
&+ 4 \sum_{i=1}^{M} \Big( \int_D \sigma(w_i) : \varepsilon(v - w_0)\, dx \Big)^2 \Big\} + \Big( \int_D \sigma(v - w_0) : \varepsilon(v - w_0)\, dx \Big)^2 \\
= {}& \mathbb{E}^2[J(S;\omega)] + \Big( \mathbb{E}[Y^4] - 1 \Big) \sum_{i=1}^{M} \Big( \int_D \sigma(w_i) : \varepsilon(w_i)\, dx \Big)^2 \\
&- 4\mathbb{E}[Y^3] \sum_{i=1}^{M} \Big( \int_D \sigma(w_i) : \varepsilon(w_i)\, dx \Big) \Big( \int_D \sigma(w_i) : \varepsilon(v - w_0)\, dx \Big)^2 \\
&+ 2 \sum_{i \neq j}^{M} \Big( \int_D \sigma(w_i) : \varepsilon(w_j)\, dx \Big)^2 + 4 \sum_{i=1}^{M} \Big( \int_D \sigma(w_i) : \varepsilon(v - w_0)\, dx \Big)^2
\end{aligned}
$$

Next from (6.1) and (4.1) we obtain

$$
\begin{aligned}
\mathbb{V}[J(S;\omega)] = & \Big( \mathbb{E}[Y^4] - 3 \Big) \sum_{i=1}^{M} \Big( \int_D \sigma(w_i) : \varepsilon(w_i)\, dx \Big)^2 \\
&- 4\mathbb{E}[Y^3] \sum_{i=1}^{M} \Big( \int_D \sigma(w_i) : \varepsilon(w_i)\, dx \Big) \Big( \int_D \sigma(w_i) : \varepsilon(v - w_0)\, dx \Big)^2 \\
&+ 2 \sum_{i,j=1}^{M} \Big( \int_D \sigma(w_i) : \varepsilon(w_j)\, dx \Big)^2 + 4 \sum_{i=1}^{M} \Big( \int_D \sigma(w_i) : \varepsilon(v - w_0)\, dx \Big)^2
\end{aligned}
\tag{4.2}
$$

Finally from Green's formula we get expression for variance.

## 6.3 Proof of Lemma 3.

We begin with the observation that for $y = \Psi_t(x)$ we have

$$
\begin{aligned}
&\int_D \sigma_t(u_t(y)) : \varepsilon(h_t(y))\, dy = \\
&\frac{1}{4} \int_D \big( \mathbb{C}(x(y))(\nabla_y u_t(y) + (\nabla_y u_t(y))^\top) \big) : \big( \nabla_y h_t(y) + (\nabla_y h_t(y))^\top \big)\, dy
\end{aligned}
$$

After change of variables $y \to x$ we obtain

$$
\int_D \sigma_t(u_t(y)) : \varepsilon(h_t(y))\, dy =
$$

$$\frac{1}{4}\int_D \big(\mathbb{C}(\nabla_y u_t(y(x)) + (\nabla_y u_t(y(x)))^\top)\big) : \big(\nabla_y h_t(y(x)) + (\nabla_y h_t(y(x)))^\top\,\big)\,\frac{\partial y}{\partial x}\,dx. \quad (4.3)$$

Notice that

$$U_t(x) = u_t(y(x)), \quad H_t(x) = h_t(y(x)),$$

which yields

$$\nabla_x U_t = \nabla_y u_t(y(x))\nabla_x y(x) = \nabla_y u_t(y(x))M(x)$$
$$\nabla_x H_t = \nabla_y h_t(y(x))\nabla_x y(x) = \nabla_y h_t(y(x))M(x),$$

or

$$\nabla_y u_t(y(x)) = \nabla_x U_t M(x)^{-1}, \quad \nabla_y h_t(y(x)) = \nabla_x H_t M(x)^{-1},$$

where

$$M(x) = \nabla_x y(x) = \nabla\Psi_t(x) = I + t\nabla\theta(x), \quad \frac{\partial y}{\partial x} = \det M.$$

Substituting obtained formula into (4.3) we arrive at the identity

$$\int_D \sigma_t(u_t(y)) : \varepsilon(h_t(y))\,dy =$$
$$\frac{1}{4}\int_D \big(\mathbb{C}(\nabla U_t M^{-1} + (\nabla U_t M^{-1})^\top)\big) : \big(\nabla H_t M^{-1} + (\nabla H_t M^{-1})^\top\,\big)\,\det M\,dx. \quad (4.4)$$

It is easily seen that

$$\partial_t M(x)\Big|_{t=0} = \nabla\theta(x), \quad \partial_t M^{-1}(x)\Big|_{t=0} = -\nabla\theta(x), \quad \partial_t \det M(x)\Big|_{t=0} = \operatorname{div}\,\theta. \quad (4.5)$$

Differentiating both sides of (4.4) with respect to $t$ at point $t = 0$ and using the relations (3.1) and (4.5) we arrive at desired identity (3.2).

### 6.4 Computation of the volume expression of the shape gradient. Proof of Theorem 3.

Let $v_t, w_t \in H^1_D(S_t)$ be solutions to the boundary value problems (2.9) corresponding to body with inclusion $S_t$. Then the Kohn-Vogelius functional $J(S_t)$ is defined by the equality

$$J(S_t) = \int_D \sigma_t(v_t - w_t) : \varepsilon(v_t - w_t)\,dx. \quad (4.6)$$

Let us set as before

$$u_t = v_t - w_t, \quad U_t = V_t - W_t, \quad V_t = v_t \circ \Psi_t, \quad W_t = w_t \circ \Psi_t,$$
$$U' = V' - W', \quad V' = \partial_t V_t|_{t=0}, \quad W' = \partial_t W_t|_{t=0},$$
$$u = v - w = u_0 = v_0 - w_0, \quad \mathbb{C} = \mathbb{C}_0, \quad S = S_0.$$

Substituting these equalities in identity (3.2) we obtain

$$dJ(S)\langle\theta\rangle = \int_D \big(\sigma(V'-W') - \mathbb{C}E(v-w,\theta) + \operatorname{div}\theta\,\sigma(v-w)\big) : \varepsilon(v-w)\,dx$$
$$+ \int_D \sigma(v-w) : \big(\varepsilon(V'-W') - E(v-w,\theta)\big)\,dx, \tag{4.7}$$

Since the coefficient tensor $\mathbb{C}$ is symmetric, we have

$$\int_D \sigma(v-w) : \big(\varepsilon(V'-W') - E(v-w,\theta)\big)\,dx =$$
$$\int_D \big(\sigma(V'-W') - \mathbb{C}E(v-w,\theta)\big) : \varepsilon(v-w)\,dx$$

Substituting this equality in (4.7) we obtain

$$dJ(S)\langle\theta\rangle = 2\int_D \big(\sigma(V'-W') - \mathbb{C}E(v-w,\theta) + \frac{1}{2}\operatorname{div}\theta\,\sigma(v-w)\big) : \varepsilon(v-w)\,dx. \tag{4.8}$$

Our next task is to eliminate $V'$ and $W'$. To this end we use equalities (3.6) and (3.8), setting $\varphi = v - w$ we obtain and subtracting one from the other we get

$$\begin{aligned}\int_D \sigma(V'-W') : \varepsilon(v-w)\,dx \\ = -\int_D \big(\mathbb{C}E(v-w,\theta) - \operatorname{div}\theta\,\sigma(v-w)\big) : \varepsilon(v-w)\,dx + \\ + \int_D \mathbf{b}(v-w,\theta) : \nabla(v-w)\,dx - \int_{\Gamma_N} (\sigma(W')\mathbf{n})\cdot(v-w)\,ds\end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.8) we obtain

$$\begin{aligned} dJ(S)\langle\theta\rangle &= 2\int_D \big(\mathbb{C}E(v-w,\theta) - \operatorname{div}\theta\,\sigma(v-w)\big) : \varepsilon(v-w)\,dx \\ &\quad +2\int_D \mathbf{b}(v-w,\theta) : \nabla(v-w)\,dx - 2\int_{\Gamma_N} (\sigma(W')\mathbf{n})\cdot(v-w)\,ds \\ &\quad +2\int_D \big(-\mathbb{C}E(v-w,\theta) + \frac{1}{2}\operatorname{div}\theta\,\sigma(v-w)\big) : \varepsilon(v-w)\,dx. \end{aligned}$$

which gives

$$dJ(S)\langle\theta\rangle = 2\int_D \mathbf{b}(v-w,\theta) : \nabla(v-w)\,dx -$$
$$\int_D \operatorname{div}\theta\,\sigma(v-w) : \varepsilon(v-w)\,dx - 2\int_{\Gamma_N} (\sigma(W')\mathbf{n})\cdot(v-w)\,ds. \tag{4.10}$$

Next we use the following formula

$$\int_{\Gamma_N} (\sigma(\phi)\mathbf{n})\cdot\psi\,ds - \int_{\Gamma_N} (\sigma(\psi)\mathbf{n})\cdot\phi\,ds = \int_D (\operatorname{div}\sigma(\phi))\cdot\psi\,dx - \int_D (\operatorname{div}\sigma(\psi))\cdot\phi\,dx$$

$$\phi = v - w, \quad \psi = W'$$

and note that

$$W' = 0 \text{ on } \Gamma, \quad \text{div } \sigma(v-w) = 0.$$

We get from this that

$$\int_{\Gamma_N} (\sigma(W')\mathbf{n}) \cdot (v-w)\, ds = \int_D (\text{ div } \sigma(W')) \cdot (v-w)\, dx$$

It follows from (3.8) and Green's formula that div $\sigma(W') =$ div $\mathbf{A}$. Thus we get

$$\int_{\Gamma_N} (\sigma(W')\mathbf{n}) \cdot (v-w)\, ds = \int_D \text{ div } \mathbf{A} \cdot (v-w)\, dx$$

From (3.11), we obtain that

$$-\int_{\Gamma_N} (\sigma(W')\mathbf{n}) \cdot (v-w)\, ds = \int_D (\mathbb{C}E(w,\theta) - \text{div } \theta\sigma(w)) : \varepsilon(v-w)\, dx + \int_D \mathbf{b}(w,\theta) : \nabla(v-w)\, dx.$$

Substituting this equality in (4.10), it holds

$$\begin{aligned} dJ(S)\langle\theta\rangle = {} & 2\int_D \mathbf{b}(v-w,\theta) : \nabla(v-w)\, dx - \int_D \text{ div } \theta\, \sigma(v-w) : \varepsilon(v-w)\, dx + \\ & 2\int_D (\mathbb{C}_0 E(w,\theta) - \text{div } \theta\, \sigma(w)) : \varepsilon(v-w)\, dx + 2\int_D \mathbf{b}(w,\theta) : \nabla(v-w)\, dx = \\ & 2\int_D \mathbf{b}(v,\theta) : \nabla(v-w)\, dx - \int_D \text{ div } \theta\, \sigma(v) : \varepsilon(v)\, dx + \\ & \int_D \text{ div } \theta\, \sigma(w) : \varepsilon(w)\, dx + 2\int_D \mathbb{C}_0 E(w,\theta) : \varepsilon(v-w)\, dx. \end{aligned}$$

Noting that

$$\int_D \mathbf{b}(v,\theta) : \nabla(v-w) = \int_D \mathbb{C}E(v-w,\theta) : \varepsilon(v)$$

we obtain

$$dJ(S)\langle\theta\rangle = 2\int_D \mathbb{C}E(v,\theta) : \varepsilon(v)\, dx - 2\int_D \mathbb{C}E(w,\theta) : \varepsilon(w)\, dx - \int_D \text{div } \theta \left( \sigma(v) : \varepsilon(c) - \sigma(w) : \varepsilon(w) \right) dx$$

or equivalently

$$dJ(S)\langle\theta\rangle = 2\int_D \sigma(v) : E(v,\theta)\, dx - 2\int_D \sigma(w) : E(w,\theta)\, dx - \int_D \text{div } \theta \left( \sigma(v) : \varepsilon(v) - \sigma(w) : \varepsilon(w) \right) dx.$$

## 6.5 Proof of the surface expression of shape gradient.

**Step 1.** We have in $D \setminus \Sigma$

$$\sigma(v) : E(v,\theta) = \frac{1}{2}\big(\partial_j(\theta_k \partial_k v_i \sigma_{ij}) + \partial_i(\theta_k \partial_k v_j \sigma_{ij})\big) - \frac{1}{2}\theta_k \partial_k(\sigma(v) : \varepsilon(v)) \tag{4.11}$$

and

$$\sigma(w) : E(w,\theta) = \frac{1}{2}\big(\partial_j(\theta_k \partial_k w_i \sigma_{ij}) + \partial_i(\theta_k \partial_k w_j \sigma_{ij})\big) - \frac{1}{2}\theta_k \partial_k(\sigma(w) : \varepsilon(w)). \tag{4.12}$$

Integrating both sides of (4.11) and (4.12) over $D \setminus \Sigma$ we obtain

$$\int_D \big(\sigma(v) : E(v,\theta) - \sigma(w) : E(w,\theta)\big)\, dx = -\frac{1}{2}\int_\Sigma \big(n_j \theta_k [\partial_k v_i \sigma_{ij}(v) - \partial_k w_i \sigma_{ij}(w))]\big)\, ds - \frac{1}{2}\int_\Sigma \big(n_i \theta_k [\partial_k v_j \sigma_{ij}(v) - \partial_k w_j \sigma_{ij}(w))]\,\big)\, ds - \frac{1}{2}\int_D \theta_k \big(\partial_k(\sigma(v) : \varepsilon(v)) - \partial_k(\sigma(w) : \varepsilon(w)\,\big) dx. \tag{4.13}$$

Notice that

$$\big(n_j \theta_k [\partial_k v_i \sigma_{ij}(v) - \partial_k w_i \sigma_{ij}(w))]\big) + \big(n_i \theta_k [\partial_k v_j \sigma_{ij}(v) - \partial_k w_j \sigma_{ij}(w))]\,\big) =$$
$$2\theta_k[\partial_k(v) \cdot (\sigma(v)n)] - 2\theta_k[\partial_k(w) \cdot (\sigma(w)n)] = 2[(\nabla v\, \theta) \cdot (\sigma(v)n) - (\nabla w\, \theta) \cdot (\sigma(w)n)].$$

Substituting this result in (4.13) we arrive at the equality

$$\int_D \big(\sigma(v) : E(v,\theta) - \sigma(w) : E(w,\theta)\big)\, dx \tag{4.14}$$
$$= -\int_\Sigma [(\nabla v\theta) \cdot (\sigma(v)n) - (\nabla w\theta) \cdot (\sigma(w)n)] ds\; - \frac{1}{2}\int_D \theta_k \big(\partial_k(\sigma(v) : \varepsilon(v)) - \partial_k(\sigma(w) : \varepsilon(w))\,\big) dx.$$

**Step 2.** Next, consider the integral

$$\int_D \operatorname{div}\,\theta\, \big(\, \sigma(v) : \varepsilon(v) - \sigma(w) : \varepsilon(w)\,\big)\, dx. \tag{4.15}$$

We have

$$\operatorname{div}\,\theta\, \big(\, \sigma(v) : \varepsilon(v) - \sigma(w) : \varepsilon(w)\,\big)\, dx =$$
$$\partial_k\big(\, \theta_k\big(\, \sigma(v) : \varepsilon(v) - \sigma(w) : \varepsilon(w))\,\big) - \theta_k \partial_k\big(\, \sigma(v) : \varepsilon(v) - \sigma(w) : \varepsilon(w)\,\big),$$

Substituting this equality in (4.15) and integrating by parts we obtain

$$\int_D \operatorname{div}\,\theta\, \big(\, \sigma(v) : \varepsilon(v) - \sigma(w) : \varepsilon(w)\,\big)\, dx = \tag{4.16}$$
$$-\int_\Sigma [\sigma(v) : \varepsilon(v) - \sigma(w) : \varepsilon(w)]\, \theta \cdot n\, ds\; \int_D \theta_k\, \partial_k\big(\, \sigma(v) : \varepsilon(v) - \sigma(w) : \varepsilon(w)\,\big)\, dx.$$

**Step 3.** Substituting expressions (4.14) and (4.16) into (3.12) we obtain

$$dJ(S)\langle\theta\rangle = \int_\Sigma [\sigma(v) : \varepsilon(v) - \sigma(w) : \varepsilon(w)]\, \theta\cdot n\, ds - 2\int_\Sigma [(\nabla v\theta)\cdot(\sigma(v)\mathbf{n}) - (\nabla w\theta)\cdot(\sigma(w)n)] ds. \tag{4.17}$$

Noticing thanks to regularity assumption that $\sigma(v)n$ and $\sigma(w)n$ have a continuous trace on $\Sigma$ we obtain

$$\int_\Sigma [(\sigma(v)n)\cdot(\nabla v\theta)]\, ds = \int_\Sigma (\sigma(v)n)\cdot[\nabla v\theta]\, ds = \int_\Sigma (\sigma(v)n)\cdot[\partial_n(v)]\theta_n\, ds,$$

and

$$\int_\Sigma [(\sigma(w)n)\cdot(\nabla w\theta)]\, ds = \int_\Sigma (\sigma(w)n)\cdot[\nabla w\theta]\, ds = \int_\Sigma (\sigma(w)n)\cdot[\partial_n(w)]\theta_n\, ds.$$

Substituting expressions into (4.17) we finally obtain

$$dJ(S)\langle\theta\rangle = \int_\Sigma [\sigma(v) : \varepsilon(v) - \sigma(w) : \varepsilon(w)]\, \theta\cdot n\, ds - 2\int_\Sigma \big((\sigma(v)n)\cdot[\partial_n(v)] - (\sigma(w)n)\cdot[\partial_n(w)]\big)\theta\cdot n\, ds.$$

## 6.6 Computation of the shape gradient of the variance. Proof of proposition 5.

We will consider variance in following form

$$\mathbb{V}[J(S;\omega)] = 2\sum_{i,j=1}^{M}\left(\int_D \sigma(w_i) : \varepsilon(w_j)\, dx\right)^2 + 4\sum_{i=1}^{M}\left(\int_D \sigma(w_i) : \varepsilon(v - w_0)\, dx\right)^2.$$

Then by chain rule we obtain

$$\begin{aligned} d(\mathbb{V}[J(S;\omega)])\langle\theta\rangle = {}& 4\sum_{i,j=1}^{M}\left(\int_D \sigma(w_i) : \varepsilon(w_j)\, dx\right)\frac{d}{dt}\left(\int_D \sigma(w_i) : \varepsilon(w_j)\, dx\right) \\ & + 8\sum_{i=1}^{M}\left(\int_D \sigma(w_i) : \varepsilon(v - w_0)\, dx\right)\frac{d}{dt}\left(\int_D \sigma(w_i) : \varepsilon(v - w_0)\, dx\right). \end{aligned}$$

Following the proof of the Theorem 3 we obtain for every $i, j = 1..M$

$$\begin{aligned} \frac{d}{dt}\left(\int_D \sigma(w_i) : \varepsilon(w_j)\, dx\right)\langle\theta\rangle &= 2\int_D \sigma(w_i) : E(w_j,\theta)\, dx - \int_D \operatorname{div}\theta\, \sigma(w_i) : \varepsilon(w_j)\, dx \\ &= \int_\Sigma [\sigma(w_i) : \varepsilon(w_j)]\, \theta\cdot n\, ds - 2\int_\Sigma (\sigma(w_i)n)\cdot[\partial_n(w_j)]\theta\cdot n\, ds \end{aligned}$$

and similarly

$$\frac{d}{dt}\left(\int_D \sigma(w_i) : \varepsilon(v - w_0)\, dx\right)\langle\theta\rangle$$

$$= 2\int_D \sigma(w_i) : E(v,\theta)\,dx - 2\int_D \sigma(w_i) : E(w_0,\theta)\,dx \;-\int_D \mathrm{div}\,\theta\,\big(\sigma(w_i) : \varepsilon(v) - \sigma(w_i) : \varepsilon(w_0)\big)\,dx$$
$$= \int_\Sigma [\sigma(w_i) : \varepsilon(v) - \sigma(w_i) : \varepsilon(w_0)]\,\theta\cdot n\,ds \;- 2\int_\Sigma \big((\sigma(w_i)n)\cdot[\partial_n(v)] - (\sigma(w_i)n)\cdot[\partial_n(w_0)]\big)\theta\cdot n\,ds.$$

Then, we get the final expression by Green's formula.

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