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Stability of Steady States in Ferromagnetic Rings

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In this paper we consider a one-dimensional model of ferromagnetic rings taking into account curvature and anisotropy effects. We describe relevant stationary configurations of the magnetization and we investigate their stability in the Liapunov sense.

I. INTRODUCTION

Ferromagnetic nanowires give promising solutions to design low-power-consuming devices ensuring non-volatile storage with fast access to the information (see [20]). In such devices, one observes the formation of domains in which the magnetization is almost constant and tangent to the wire direction. These domains are separated by domain walls (DWs), thin zones in which the magnetization switches abruptly. This property plays an important role in numerical data storage since the information is encoded by the localization of DWs (see [2, 15, 20]). Because of these potential applications, DW dynamics in ferromagnetic wires is intensively investigated as well in Physics (see [16, 19, 23–27]) as in Mathematics (see [3, 8, 10, 17, 22]). In [12, 14, 18], DW dynamics and magnetization reversal in ferromagnetic rings are investigated. In the present paper, we consider a one-dimensional model of ferromagnetic ring justified by asymptotic methods in [4]. We aim to exhibit all the in-plane stationary solutions and to investigate their stability. In [17], S. Labbé, Y. Privat, and E. Trélat address a similar issue: existence and stability of L-periodic steady states for an infinite straight nanowire. They compute all these configurations and prove their instability, excepted for uniform configurations in which the magnetic moment is oriented in the direction of the wire. Here, we enrich the model by taking into account curvature and anisotropy effects due to the geometry of the considered rings. The consequence is that we exhibit more stable configurations, which reflects the complex landscape observed in experiments.

A. Model for ferromagnetic rings

We recall the usual 3-dimensional dynamical model of ferromagnetism.

We denote by (e_1, e_2, e_3) the canonical basis of $\mathbb{R}^3$, by $\cdot$ (resp. $|.|$) the euclidean scalar product (resp. the euclidean norm) in $\mathbb{R}^3$, and by $\times$ the cross product. We consider the ring Ω_η obtained by rotation around the $\mathbf{z}$-axis of the ellipse contained in the plane $\mathbf{x} = 0$ of equation $\frac{(\mathbf{y}-\mathbf{R})^2}{a^2} + \frac{\mathbf{z}^2}{b^2} < \eta^2$, where $a > b$ and η is a small dimensionless parameter.

Ferromagnetic materials are characterised by a spontaneous magnetization described by a vector field called magnetic moment. We consider a ferromagnetic device occupying the ring Ω_η. We denote by $\mathbf{M}(\mathbf{t},\mathbf{X})$ the distribution of the magnetization at time $\mathbf{t}$ and at point $\mathbf{X} = (\mathbf{x},\mathbf{y},\mathbf{z}) \in \Omega_\eta$. We suppose that the material is saturated, *i.e.* the norm of $\mathbf{M}$ is a constant equal to $\mathbf{M}_s$ (expressed in $\mathrm{A.m}^{-1}$):

$$|\mathbf{M}(\mathbf{t},\mathbf{X})| = \mathbf{M}_s \text{ a.e.} \tag{1}$$

The magnetic induction $\mathbf{B}$ and the magnetic field $\mathbf{H}$ are linked by the constitutive relation

$$\mathbf{B} = \mu_0(\mathbf{H} + \overline{\mathbf{M}}),$$

where $\overline{\mathbf{M}}$ is the extension of $\mathbf{M}$ by zero outside Ω_η, and where $\mu_0 = 4\pi.10^{-7}\,\mathrm{kg.m.s^{-2}.A^{-2}}$ is the vacuum permeability.

The variations of $\mathbf{M}$ satisfy the following Landau-Lifschitz equation (see [1, 6, 13]):

$$\frac{\partial \mathbf{M}}{\partial \mathbf{t}} = -\gamma \mathbf{M} \times \mathbf{H}_{\text{eff}}(\mathbf{M}) - \frac{\alpha\gamma}{\mathbf{M}_s}\mathbf{M} \times (\mathbf{M} \times \mathbf{H}_{\text{eff}}(\mathbf{M})), \tag{2}$$

where:

- γ (expressed in $\mathrm{A.s.kg^{-1}}$) is the gyromagnetic ratio,
- the dimensionless term α is the damping coefficient,
- $\mathbf{H}_{\text{eff}}$ is the effective field (expressed in $\mathrm{T} = \mathrm{kg.s^{-2}.A^{-1}}$) given by:

$$\mathbf{H}_{\text{eff}}(\mathbf{M}) = \frac{A}{\mathbf{M}_s^2}\Delta \mathbf{M} + \mu_0 \mathbf{H}_d(\mathbf{M}),$$

  where we denote by A the exchange constant (expressed in $\mathrm{J.m^{-1}}$) and by $\mathbf{H}_d(\mathbf{M})$ the demagnetising field obtained from $\mathbf{M}$ by solving the Maxwell-Faraday equation:

$$\text{curl}\, \mathbf{H}_d(\mathbf{M}) = 0 \quad \text{and} \quad \text{div}(\mathbf{H}_d(\mathbf{M}) + \overline{\mathbf{M}}) = 0 \quad \text{in } \mathbb{R}^3. \tag{3}$$

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The effective field $\mathbf{H}_{\text{eff}}(\mathbf{M})$ is related to the micromagnetism energy $\mathscr{E}_m(\mathbf{M})$ by the relation $\mathbf{H}_{\text{eff}} = -\partial_{\mathbf{M}}\mathscr{E}_m$, with

$$\mathscr{E}_m(\mathbf{M}) = \frac{1}{2}\frac{A}{\mathbf{M}_s^2}\int_{\Omega_\eta} |\nabla_{\mathbf{X}}\mathbf{M}|^2 d\mathbf{X} + \frac{1}{2}\mu_0\int_{\mathbb{R}^3} |\mathbf{H}_d(\mathbf{M})|^2 d\mathbf{X}. \tag{4}$$

In order to obtain a dimensionless model, we write $\mathbf{M}(\mathbf{t},\mathbf{X}) = \mathbf{M}_s\mathbf{m}(\frac{\mathbf{t}}{\tau},\frac{\mathbf{X}}{\mathbf{R}})$, where the characteristic time τ is given by

$$\tau = \frac{\mathbf{R}^2\mathbf{M}_s}{\gamma A}$$

(we recall that $\mathbf{R}$ is the radius of the ring). We denote by $t = \frac{\mathbf{t}}{\tau}$ the dimensionless time and by $X = (x,y,z) = \frac{\mathbf{X}}{\mathbf{R}} \in \mathbb{R}^3$ the dimensionless position. Then, $\mathbf{m}(t,X)$ is defined for $t \in \mathbb{R}^+$ and $X \in \mathscr{O}_\eta$, where $\mathscr{O}_\eta$ is the ring obtained by rotation around the z-axis of the ellipse contained in the plane $x=0$ of equation $\frac{\mathbf{R}^2(y-1)^2}{a^2} + \frac{\mathbf{R}^2 z^2}{b^2} \leq \eta^2$. It satisfies:

$$\begin{cases} \mathbf{m} : \mathbb{R}^+ \times \mathscr{O}_\eta \longrightarrow S^2, \text{ where } S^2 \text{ is the unit sphere of } \mathbb{R}^3, \\ \dfrac{\partial \mathbf{m}}{\partial t} = -\mathbf{m}\times\mathbf{H}(\mathbf{m}) - \alpha\mathbf{m}\times(\mathbf{m}\times\mathbf{H}(\mathbf{m})), \\ \mathbf{H}(\mathbf{m}) = \Delta\mathbf{m} + \mathbf{H}_d(\mathbf{m}). \end{cases} \tag{5}$$

Using [4] and [21], we take the limit of the 3d-model (5) when η tends to zero. The limit domain is the unit circle contained in the plane $z=0$ (of equations $x^2+y^2=1$ and $z=0$). The limit m of the 3d-magnetic moment $\mathbf{m}$ only depends on time t and on the position on the circle parametrised by arc-length by $\theta \mapsto \begin{pmatrix}\cos\theta \\ \sin\theta \\ 0\end{pmatrix}$. It satisfies:

$$\begin{cases} m : \mathbb{R}_t^+ \times \mathbb{R}_\theta \longrightarrow S^2, 2\pi\text{-periodic in the variable } \theta, \\ \dfrac{\partial m}{\partial t} = -m\times\mathscr{H}_{\text{eff}}(m) - \alpha m\times(m\times\mathscr{H}_{\text{eff}}(m)), \\ \mathscr{H}_{\text{eff}}(m) = \dfrac{\partial^2 m}{\partial\theta^2} + \dfrac{1}{\lambda}\mathscr{H}_d(m), \\ \quad \text{with} \quad \mathscr{H}_d(m) = -\dfrac{b}{a+b}(m\cdot e_r)e_r - \dfrac{a}{a+b}m_3 e_3, \end{cases} \tag{6}$$

where we denote:

$$e_r = \begin{pmatrix}\cos\theta \\ \sin\theta \\ 0\end{pmatrix}, \quad e_\theta = \begin{pmatrix}-\sin\theta \\ \cos\theta \\ 0\end{pmatrix}, \quad e_3 = \begin{pmatrix}0 \\ 0 \\ 1\end{pmatrix},$$

and where the dimensionless parameter λ is given by:

$$\lambda = \frac{A}{\mu_0\mathbf{M}_s^2\mathbf{R}^2}.$$

The dimensionless energy related to the limit model is given by :

$$\mathscr{E}(m) = \frac{1}{2}\int_0^{2\pi} |\frac{\partial m}{\partial\theta}|^2 d\theta + \frac{1}{2\lambda}\int_0^{2\pi}\left(\frac{b}{a+b}(m\cdot e_r)^2 + \frac{a}{a+b}|m_3|^2\right)d\theta. \tag{7}$$

As already observed in [4, 10, 21], we remark that the limit demagnetising operator $\mathscr{H}_d$ is local in the one-dimensional model and is analogous to an anisotropic term. We remark also that (6) is invariant by rotation-translation, *i.e.* if m satisfies (6), then for all $\varphi \in \mathbb{R}$, $(t,\theta) \mapsto R_\varphi m(t,\theta-\varphi)$ is also solution for (6), with

$$R_\varphi = \begin{pmatrix}\cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1\end{pmatrix}. \tag{8}$$

In what follows we define the spaces:

$$L_p^2 = \{f \in L_{loc}^2(\mathbb{R},\mathbb{R}^N), \text{ such that for a.e. } x, f(x+2\pi) = f(x)\},$$
$$H_p^k = \{w \in H_{loc}^k(\mathbb{R},\mathbb{R}^N), \text{ such that for all } x, w(x+2\pi) = w(x)\}.$$

We denote by $< \mid >$ the scalar product in L_p^2 and by $\|\ .\ \|_{L_p^2}$ the associated norm:

$$< u \mid v >= \int_0^{2\pi} u(\theta)v(\theta)d\theta, \quad \|u\|_{L_p^2} = \left(\int_0^{2\pi}|u(\theta)|^2 d\theta\right)^{\frac{1}{2}}.$$

We also denote by $\|\ .\ \|_{H_p^k}$ the norm in H_p^k:

$$\|u\|_{H_p^k} = \left(\sum_{j=0}^{k}\left\|\frac{d^j u}{d\theta^j}\right\|_{L_p^2}^2\right)^{\frac{1}{2}}.$$

B. Statement of the main results

Our goal is to describe all the in-plane steady states of (6) and to discuss their stability. The in-plane solutions are of particular interest in applications. Indeed, in experimental devices, the considered rings are thin in the $\mathbf{z}$-direction, so that $a > b$ in our model. Since the demagnetising field behaves like an anisotropic term where the z-axis is a bad axis of magnetization, the observed configurations in the experiments are in-plane.

We write the in-plane steady states M^0 for (6) on the form:

$$M^0(\theta) = (\cos u(\theta))e_r(\theta) + (\sin u(\theta))e_\theta(\theta), \tag{9}$$

where $u \in H_{loc}^1(\mathbb{R};\mathbb{R})$ satisfies:

$$\exists k \in \mathbb{Z}, \forall\theta \in \mathbb{R}, u(\theta+2\pi) = u(\theta) + 2k\pi. \tag{10}$$

The latest condition in necessary to ensure that M^0 is 2π-periodic. We remark that $k+1$ is the topological degree of

M^0 as a function from the unit circle S^1 into itself. In Section II, we calculate all these static solutions. A simple calculation gives that M^0 satisfies (6) if and only if u satisfies the pendulum equation:

$$u'' + \frac{b}{\lambda(a+b)} \cos u \sin u = 0, \tag{11}$$

and we use a shooting method to construct solutions for the pendulum equation satisfying (10). In the case $k = 0$, we establish the following theorem:

Theorem 1. *Let $\lambda > 0$, $a > 0$ and $b > 0$. Let $l \in \mathbb{N}$ such that $l < \sqrt{\frac{b}{\lambda(a+b)}} \leq l+1$. There exists l distinct 2π-periodic solutions $\mathscr{M}_j^0$ for (6), which topological degree equals one, and such that all degree-one steady state M^0 is equal to either $\pm e_\theta$, $\pm e_r$ or $\pm R(\varphi)\mathscr{M}_j^0(\theta - \varphi)$, where $\varphi \in \mathbb{R}$.*

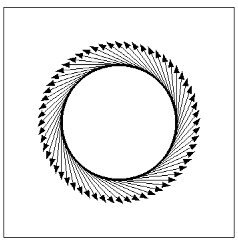

FIG. 1. Solution e_θ.

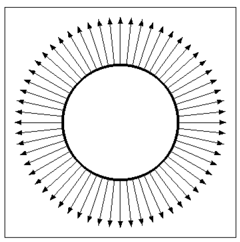

FIG. 2. Solution e_r

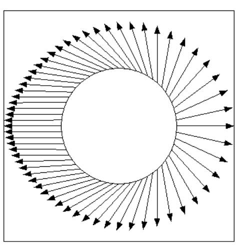

FIG. 3. Solution $\mathscr{M}_1^0$

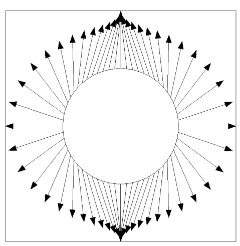

FIG. 4. Solution $\mathscr{M}_2^0$.

For $k \neq 0$, the landscape is simpler, since we prove existence and uniqueness up to translation-rotation of the solutions:

Theorem 2. *Let $\lambda > 0$, let $k \in \mathbb{Z}$, $k \neq 0$. There exists a stationary solution for (6) denoted $M_{k,\lambda}^0$, of degree $k+1$, such that all steady state M^0 of degree $k+1$ satisfies:*

$$\exists \varphi \in \mathbb{R}, \forall \theta \in \mathbb{R}, M^0(\theta) = R(\varphi) M_{k,\lambda}^0(\theta - \varphi).$$

We investigate also the stability of the solutions described in the previous Theorems. We recall that a steady-state solution $M^0 \in H_p^1$ is said to be stable for (6) if for all $\varepsilon > 0$, there exists $\eta_0 > 0$ such that all solution m of (6) such that $\|m(0,\cdot) - M^0(\cdot)\|_{H_p^1} \leq \eta_0$ satisfies:

$$\forall t > 0, \|m(t,\cdot) - M^0(\cdot)\|_{H_p^1} \leq \varepsilon.$$

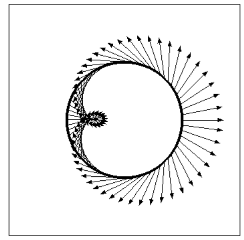

FIG. 5. Solution $M_{1,\lambda}^0$.

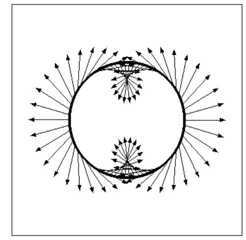

FIG. 6. Solution $M_{2,\lambda}^0$.

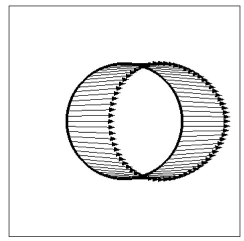

FIG. 7. Solution $M_{-1,\lambda}^0$.

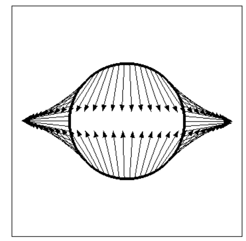

FIG. 8. Solution $M_{-2,\lambda}^0$.

In addition, we say that $M^0 \in H_p^1$ is asymptotically stable if it is stable and if there exists $\eta_1 > 0$ such that all solution m for (6) such that $\|m(0,\cdot) - M^0(\cdot)\|_{H_p^1} \leq \eta_1$ satisfies:

$$\|m(t,\cdot) - M^0(\cdot)\|_{H_p^1} \to 0 \text{ when } t \to +\infty.$$

Concerning the solutions in the case $k = 0$, we first obtain the instability of $\pm e_r$:

Theorem 3. *Let $a > 0$ and $b > 0$ such that $a > b$. For all $\lambda > 0$, $\pm e_r$ is unstable.*

Concerning $\pm e_\theta$, we establish the following result:

Theorem 4. *Let $a > 0$ and $b > 0$ such that $a > b$. If $\lambda > \frac{a}{a+b}$, then $\pm e_\theta$ is unstable and if $\lambda < \frac{a}{a+b}$, then $\pm e_\theta$ is asymptotically stable.*

In the critical case $\lambda = \frac{a}{a+b}$, we remark that there exists a one-parameter family of steady states given by:

$$M_\varphi : \theta \mapsto \cos\varphi e_\theta(\theta) + \sin\varphi e_3.$$

We obtain the following stability theorem for e_θ:

Theorem 5. *Let $a > 0$ and $b > 0$ such that $a > b$. We assume that $\lambda = \frac{a}{a+b}$. Then e_θ is stable. In addition, there exists $\eta_1 > 0$ such that all solution m for (6) such that $\|m(0,\cdot) - e_\theta\|_{H_p^1} \leq \eta_1$ satisfies that there exists $\varphi \in \mathbb{R}$ such that $m(t,\cdot)$ tends to M_φ in H_p^1 when t tends to $+\infty$.*

We conclude the case of the steady-state solutions of degree one by the following result:

Theorem 6. *Let $a > 0$ and $b > 0$ such that $a > b$. For all $\lambda > 0$, the solutions $\mathscr{M}_j^0$ given by Theorem 1 are linearly unstable.*

In the case $k \neq 0$, we must take into account the invariance of (6) by rotation-translation. We give the following definition:

Definition 1. *We say that $M^0 \in H^1_p$ is asymptotically stable modulo rotation-translation if it is stable and if there exists $\eta_1 > 0$ such that all solution m for (6) such that $\|m(0,\cdot) - M^0(\cdot)\|_{H^1_p} \leq \eta_1$ satisfies:*

$$\exists\, \varphi_\infty \in \mathbb{R},\ \|m(t,\cdot) - R(\varphi_\infty)M^0(\cdot - \varphi_\infty)\|_{H^1_p} \to 0 \text{ when } t \to +\infty.$$

We establish the following theorem, in which for a fixed $k \neq 0$, we discuss the stability of $M^0_{k,\lambda}$ depending on the values of λ:

Theorem 7. *Let $a > 0$ and $b > 0$ such that $a > b$. For all $k \in \mathbb{Z}^*$, for $\lambda > 0$ small enough, $M^0_{k,\lambda}$ is asymptotically stable modulo rotation-translation.*

For all k in $\mathbb{Z}\setminus\{0,-1\}$. Then, for $\lambda > 0$ large enough, $M^0_{k,\lambda}$ is linearly unstable.

We also prove the following result, in which for a fixed λ, we discuss the stability of $M^0_{k,\lambda}$ depending on k:

Theorem 8. *Let $a > 0$ and $b > 0$ such that $a > b$. Let $\lambda > 0$. Then for $|k|$ large enough, $M^0_{k,\lambda}$ is linearly unstable.*

The paper is organised as follows: Theorems 1 and 2 are proved in Section II. As already said, we use a shooting method to construct solutions of the pendulum equation (11) satisfying the degree condition (10).

The rest of the paper is devoted to the study of the stability for the solutions described in Theorems 1 and 2.

The first difficulty to be tackled when proving stability or instability results is due to the saturation constraint $|m| = 1$. In order to deal with perturbations satisfying this constraint, as in [7, 10, 11], we express the perturbations m of M^0 in the mobile frame (M^0, M^1, M^2) writing:

$$\begin{aligned} m(t,\theta) = {} & r_1(t,\theta)M^1(\theta) + r_2(t,\theta)M^2 \\ & + \sqrt{1-(r_1(t,\theta))^2-(r_2(t,\theta))^2}\,M^0(\theta), \end{aligned}$$

where:

$$M^1(\theta) = -(\sin u(\theta))e_r(\theta) + (\cos u(\theta))e_\theta(\theta) \text{ and } M^2 = e_3.$$

In Section III, we rewrite Equation (6) in the mobile frame. We obtain an equivalent equation in the new unknown $r = (r_1, r_2)$ of the form:

$$\partial_t r = \begin{pmatrix} -\alpha & -1 \\ 1 & -\alpha \end{pmatrix} \mathscr{L} r + F(\theta, r, \partial_\theta r, \partial_{\theta\theta} r), \qquad (12)$$

where $\mathscr{L} = \begin{pmatrix} -\partial_{\theta\theta} r_1 + f_1 r_1 \\ -\partial_{\theta\theta} r_2 + f_2 r_2 \end{pmatrix}$ (the functions f_1 and f_2 depend on the considered solution M^0), and where F is a non linear term, also depending on M^0. The stability of M^0 for (6) is equivalent to the stability of the null solution for (12). It is strongly related to the sign of the eigenvalues of $\mathscr{L}$.

In Section IV, we establish the instability of e_r. In this case, the coefficients of (12) do not depend on θ, so by considering a constant-in-θ perturbation of 0, we are back to the o.d.e. case, and we conclude by proving that the linearization for the o.d.e admits an eigenvalue which real part has the bad sign.

Theorems 4 and 5 are proved in Section V. The instability for large λ is established as in Section IV. For $\lambda < \frac{a}{a+b}$, $\mathscr{L}$ is positive definite and we establish the asymptotic stability by variational estimates.

In the critical case $\lambda = \frac{a}{a+b}$, $\mathscr{L}$ is positive, but 0 is in its spectrum. This is due to the fact that there is a one-parameter family $\varphi \mapsto M(\varphi)$ of steady states such that $e_\theta = M(0)$. By projection on the mobile frame, 0 belongs also to a one-parameter family $\varphi \mapsto \mathbf{V}(\varphi)$ of solutions for (12). The perturbations r of zero are then described as

$$r(t,\cdot) = \mathbf{V}(\varphi(t)) + w(t,\cdot), \text{ where } w(t,\cdot) \in (\text{Ker}\,\mathscr{L})^\perp.$$

Since $\mathscr{L}$ is positive definite on (Ker $\mathscr{L})^\perp$, variational estimates yield that $w(t,\cdot)$ tends exponentially to 0, which implies that $\varphi(t)$ tends to a finite limit φ_∞, so that r tends to $\mathbf{V}(\varphi_\infty)$.

Section VI is devoted to the proof of the linear instability for the steady states $\mathscr{M}_j^0$. We prove first that $\mathscr{L}$ admits a non-positive eigenvalue, and we establish that this implies that the null solution is unstable for the linearization of (12).

In the last section, we tackle the case of the solutions $M^0_{k,\lambda}$ given by Theorem 2. Because of the invariance by rotation-translation of Equation (6), 0 is the spectrum of $\mathscr{L}$ and we conclude with the method used for e_θ in the critical case. When $\mathscr{L}$ admits non-positive eigenvalues, we establish linear instability with the same arguments as in Section VI.

II. EXISTENCE OF STEADY-STATES SOLUTIONS

We look for in-plane steady-state solutions for (6), writing it on the form:

$$M^0(\theta) = (\cos u(\theta))e_r(\theta) + (\sin u(\theta))e_\theta(\theta). \qquad (13)$$

These solutions satisfy:

$$M^0 \times \mathscr{H}_{\text{eff}}(M^0) = 0, \qquad (14)$$

and plugging (13) into (14), we obtain that M^0 satisfies (14) if and only if u satisfies:

$$\begin{cases} u'' + \dfrac{b}{\lambda(a+b)} \cos u \sin u = 0, \\ \exists k \in \mathbb{Z}, \forall \theta \in \mathbb{R}, u(\theta + 2\pi) = u(\theta) + 2k\pi. \end{cases} \qquad (15)$$

As already said, the latest condition ensures that M^0 is 2π-periodic. By multiplying (15) by u' and by integration we obtain that there exists a constant ρ such that for all θ,

$$(u')^2 + \frac{b}{\lambda(a+b)} \sin^2 u = \rho^2. \qquad (16)$$

A. Proof of Theorem 1: case $k=0$

We look for 2π-periodic solutions u of (15). We remark first that the constant functions $\theta \mapsto k\frac{\pi}{2}$, $k \in \mathbb{Z}$, are solutions. The corresponding solutions for (6) are on the form : $\theta \mapsto \pm e_r$ for k even and $\theta \mapsto \pm e_\theta$ for k odd. If λ is small enough, there are also non constant solutions. The 2π-periodic solution for (15) are either on the separatrix, *i.e.* the lines $p = \pm \frac{b}{\lambda(a+b)} \cos u$ in the phase portrait, or inside one cell $\mathscr{C}_m$ between the separatrix, where:

$$\mathscr{C}_m = \Big\{ (u,p) \in \mathbb{R}^2,\ |u - m\pi| < \frac{\pi}{2},$$

$$|p|^2 < \frac{b}{\lambda(a+b)} |\cos u|^2 \Big\},$$

(from classical argument, a trajectory cannot cross the separatrix so it cannot get out of a cell).

In the first case, we have $\rho^2 = \frac{b}{\lambda(a+b)}$, and the only 2π-periodic solutions on the separatrix are the constant maps $\theta \mapsto u(\theta) = \frac{\pi}{2} \mod (\pi)$. Hence, the corresponding solutions for (6) are $M^0 = \pm e_\theta$.

For solutions inside the separatrix, we investigate first the solutions contained in the cell $\mathscr{C}_0$.

By translation in the variable θ, we can assume that $u(0) \in [0, \frac{\pi}{2}[$ and $u'(0) = 0$. If $u(0) = 0$, we recover the constant solution $u \equiv 0$ corresponding to $M^0 = e_r$.

In order to investigate the non constant 2π-periodic solution for (15) which trajectories are contained in $\mathscr{C}_0$, for $\gamma \in]0, \frac{\pi}{2}[$, we denote by u_γ the solution of:

$$\begin{cases} u_\gamma'' + \dfrac{b}{\lambda(a+b)} \cos u_\gamma \sin u_\gamma = 0, \\ \\ u_\gamma(0) = \gamma \quad \text{and} \quad u_\gamma'(0) = 0. \end{cases} \tag{17}$$

By classical calculation, the period $L(\gamma)$ of this solution is given by:

$$L(\gamma) = 4\sqrt{\frac{\lambda(a+b)}{b}} \int_0^\gamma \frac{du}{\sqrt{\sin^2\gamma - \sin^2 u}}. \tag{18}$$

If u_γ is $L(\gamma)$-periodic , then $nL(\gamma)$ ($n \in \mathbb{N}^*$) is also a period for u_γ. So it comes that the function u_γ satisfies $u_\gamma(0) = u_\gamma(2\pi)$ if and only if there exists $n \in \mathbb{N}^*$ such that $nL(\gamma) = 2\pi$.
The function L is continuous and non decreasing (see [11]). In addition, we have $\lim_{\gamma \to \frac{\pi}{2}} L(\gamma) = +\infty$, and $\lim_{\gamma \to 0} L(\gamma) = 2\pi\sqrt{\frac{\lambda(a+b)}{b}}$.

Therefore, if $\dfrac{b}{\lambda(a+b)} \le 1$, for all $\gamma \in]0, \frac{\pi}{2}[$, $L(\gamma) > 2\pi$, so there is no 2π-periodic solution of this type.

If $\dfrac{b}{\lambda(a+b)} > 1$, then there exists $l \in \mathbb{N}^*$ such that $l+1 \ge \sqrt{\frac{b}{\lambda(a+b)}} > l$ and we have:

$$\frac{2\pi}{l+1} \le \lim_{\gamma \to 0} L(\gamma) < \frac{2\pi}{l}.$$

So, by monotonicity arguments, for all $n \in \{1, \ldots, l\}$, there exists only one $\gamma_n \in]0, \frac{\pi}{2}[$ such that $nL(\gamma_n) = 2\pi$. Therefore, there are exactly l 2π-periodic solutions (modulo translation in θ) in the cell $\mathscr{C}_0$.

By classical argument, the solutions v contained in the cell $\mathscr{C}_1$ are on the form $\theta \mapsto \pi - u(\theta)$, where u is a solution in the cell $\mathscr{C}_0$. In addition, the solutions w in the cell $\mathscr{C}_m$ are on the form $w(\theta) = u(\theta) + m\pi$ if m is even and $w(\theta) = m\pi - u(\theta)$ if m is odd.

B. Proof of Theorem 2: case $k \neq 0$

Now we look for planar static solutions of (6) of degree $k+1$, $k \neq 0$, *i.e.* we look for solutions u for (15) such that $u(\theta + 2\pi) = u(\theta) + 2k\pi$, with $k \neq 0$. These solutions are outside the separatrix, since the solutions inside the separatrix take theirs values in intervals which sizes is less than π. These solutions satisfy (16) with $|\rho|^2 > \dfrac{b}{\lambda(a+b)}$.

For $k \ge 1$, we consider, for $\rho > \sqrt{\frac{b}{\lambda(a+b)}}$, the solution v_ρ of (15) such that $v_\rho(0) = 0$ and $v_\rho'(0) = \rho$. Writing (16), we obtain that v_ρ reaches the value $2k\pi$ at the point ρ such that:

$$\theta_k(\rho) := \int_0^{2k\pi} \frac{dv}{\sqrt{\rho^2 - \dfrac{b}{\lambda(a+b)} \sin^2 v}} = 2\pi.$$

We have: $\theta_k(\rho) = 4k \displaystyle\int_0^{\frac{\pi}{2}} \frac{dv}{\sqrt{\rho^2 - \dfrac{b}{\lambda(a+b)} \sin^2 v}}$. We remark that the function θ_k is continuous and non increasing. In addition, we have:

$$\lim_{\rho \to \sqrt{\frac{b}{\lambda(a+b)}}} \theta_k(\rho) = +\infty \text{ and } \lim_{\rho \to +\infty} \theta_k(\rho) = 0.$$

Then we deduce that for all fixed $k \ge 1$, there exist a unique $\rho \in]\sqrt{\frac{b}{\lambda(a+b)}}, +\infty[$ such that $\theta_k(\rho) = 2\pi$.
By the same way we prove the same result for $k \le -1$ with $\rho < -\sqrt{\dfrac{b}{\lambda(a+b)}}$.

III. EQUATIONS IN THE MOBILE FRAME

We investigate the stability in the Liapunov sense of the static solutions we calculated in the previous section. Since we must only consider perturbations satisfying the saturation constraint $|M| = 1$, we use the mobile frame technique introduced in [10]. Let M^0 be a static solution for Equation (6), obtained either in Theorem 1 or in Theorem 2:

$$M^0(\theta) = \cos u(\theta) e_r(\theta) + \sin u(\theta) e_\theta(\theta).$$

We denote by ρ^2 the conserved quantity:

$$\rho^2 = (u')^2 + \frac{b}{\lambda(a+b)} \sin^2 u.$$

We introduce the mobile frame $(M^0(\theta), M^1(\theta), M^2)$, where

$$M^1(\theta) = -(\sin u(\theta))e_r(\theta) + (\cos u(\theta))e_\theta(\theta) \quad \text{and} \quad M^2 = e_3.$$

We consider perturbations of M^0 as follows:

$$m(t,\theta) = r_1(t,\theta)M^1(\theta) + r_2(t,\theta)M^2 + (1+\nu(r(t,\theta)))M^0(\theta), \tag{19}$$

with $\nu(r) = \sqrt{1-r_1^2-r_2^2} - 1$, so that we consider only perturbations satisfying the saturation constraint $|m| = 1$.

We write the Landau-Lifschitz equation with this new unknown and by projection on M^1 and M^2 we obtain that if M satisfies (6), then r satisfies the following quasilinear equation:

$$\partial_t r = \Lambda r + F(\theta, r, \partial_\theta r, \partial_{\theta\theta} r). \tag{20}$$

The linear part of this equation writes $\Lambda r = J\mathscr{L} r$, where

$$J = \begin{pmatrix} -\alpha & -1 \\ 1 & -\alpha \end{pmatrix} \quad \text{and} \quad \mathscr{L} r = \begin{pmatrix} \mathscr{L}_1 r_1 \\ \mathscr{L}_2 r_2 \end{pmatrix}, \tag{21}$$

with

$$\mathscr{L}_1 = -\partial_{\theta\theta} + s_0 \text{ and } \mathscr{L}_2 = \mathscr{L}_1 + P, \tag{22}$$

where

$$\begin{aligned} s_0 &= \frac{b}{\lambda(a+b)}(\sin^2 u - \cos^2 u) \\ P &= \frac{a}{\lambda(a+b)} - (\rho^2 + 2u' + 1). \end{aligned} \tag{23}$$

The nonlinear part $F(\theta, r, \partial_\theta r, \partial_{\theta\theta} r)$ is given by:

$$\begin{aligned} F(\theta, r, \partial_\theta r, \partial_{\theta\theta} r) = \; & F_1(r)\partial_{\theta\theta} r + F_2(r)(\partial_\theta r, \partial_\theta r) + \\ & + F_3(\theta, r)\partial_\theta r + F_4(\theta, r), \end{aligned} \tag{24}$$

with

- $F_1 \in \mathscr{C}^\infty\left(B(0,\tfrac{1}{2}); \mathscr{L}(\mathbb{R}^2;\mathbb{R}^2)\right)$:

$$\begin{aligned} F_1(r) = \; & \nu(r)\begin{pmatrix} \alpha\nu(r) + 2\alpha & 1 \\ -1 & \alpha\nu(r) + 2\alpha \end{pmatrix} \\ & + \alpha \begin{pmatrix} r_2^2 & -r_1 r_2 \\ -r_1 r_2 & r_1^2 \end{pmatrix} \\ & + \begin{pmatrix} -r_2 - \alpha r_1 - \alpha r_1 \nu(r) \\ r_1 - \alpha r_2 - \alpha r_2 \nu(r) \end{pmatrix} \mathrm{d}\nu(r), \end{aligned} \tag{25}$$

- $F_2 \in \mathscr{C}^\infty\left(B(0,\tfrac{1}{2}); \mathscr{L}_2(\mathbb{R}^2 \times \mathbb{R}^2;\mathbb{R}^2)\right)$:

$$F_2(r)(\partial_\theta r, \partial_\theta r) = \begin{pmatrix} -r_2 - \alpha r_1 - \alpha r_1 \nu(r) \\ r_1 - \alpha r_2 - \alpha r_2 \nu(r) \end{pmatrix} \mathrm{d}^2\nu(r)(\partial_\theta r, \partial_\theta r), \tag{26}$$

- $F_3 \in \mathscr{C}^\infty\left(\mathbb{R} \times B(0,\tfrac{1}{2}); \mathscr{L}(\mathbb{R}^2;\mathbb{R}^2)\right)$:

$$\begin{aligned} F_3(\theta, r)\partial_\theta r = \; & 2(u'+1)\begin{pmatrix} r_2 + \alpha r_1 + \alpha r_1 \nu(r) \\ -r_1 + \alpha r_2 + \alpha r_2 \nu(r) \end{pmatrix} \partial_\theta r_1 \\ & + 2(u'+1)\begin{pmatrix} \alpha(1-(r_1)^2) \\ -1 - \nu(r) - \alpha r_1 r_2 \end{pmatrix} \mathrm{d}\nu(r)(\partial_\theta r), \end{aligned} \tag{27}$$

- $F_4 \in \mathscr{C}^\infty\left(\mathbb{R} \times B(0,\tfrac{1}{2}); \mathbb{R}^2\right)$:

$$\begin{aligned} F_4(\theta, r) = \; & 2u''(\theta)\begin{pmatrix} r_1 r_2 + \alpha r_1^2 + \alpha \nu(r) r_1^2 \\ -r_1^2 + \alpha r_1 r_2 + \alpha \nu(r) r_1 r_2 \end{pmatrix} \\ & + \alpha P(\theta)\begin{pmatrix} r_1 r_2^2 \\ -r_1^2 r_2 \end{pmatrix} \\ & - \alpha\left(\nu^2(r) + 2\nu(r)\right)\begin{pmatrix} s_0(\theta) r_1 \\ (P(\theta) + s_0(\theta)) r_2 \end{pmatrix} \\ & + \nu(r)\begin{pmatrix} -(P(\theta) + s_0(\theta)) r_2 \\ s_0(\theta) r_1 \end{pmatrix}. \end{aligned} \tag{28}$$

As it is proved in [10], using the saturation constraint $|m| = 1$ automatically satisfied from (19), we have that m satisfies (6) if and only if $r = (r_1, r_2)$ satisfies (20). In addition, M_0 is a stable solution for (6) if and only if zero is a stable solution for (20).

We remark that since ν is smooth on the open ball $B(0,1)$, Equation (20) is valid for $|r| < 1$. The nonlinear terms can be estimated as follows:

Proposition 1. *There exists K such that for all $r \in B(0,\frac{1}{2})$, for all $\theta \in \mathbb{R}$,*

$$\begin{aligned} |F_1(r)| \le K|r|^2, \; |F_2(r)| \le K|r|, \; |F_3(\theta, r)| \le K|r| \\ \text{and} \quad |F_4(\theta, r)| \le K|r|^2, \end{aligned}$$

and

$$\begin{aligned} |dF_1(r)| \le K|r|, \; |dF_2(r)| \le K, \; |dF_3(\theta, r)| \le K \\ \text{and} \quad |dF_4(\theta, r)| \le K|r|. \end{aligned}$$

(we denote by dF_j the derivative of F_j with respect to r).

IV. PROOF OF THEOREM 3

The solution e_r corresponds to $u = 0$, so Equation (20) writes

$$\partial_t r = \begin{pmatrix} -\alpha & -1 \\ 1 & -\alpha \end{pmatrix} \mathscr{L} r + G(r, \partial_\theta r, \partial_{\theta\theta} r), \tag{29}$$

where

$$\mathscr{L} = \begin{pmatrix} -\partial_{\theta\theta} - \frac{b}{\lambda(a+b)} & 0 \\ 0 & -\partial_{\theta\theta} + \frac{a-b}{\lambda(a+b)} - 1 \end{pmatrix},$$

and

$$G(r, \partial_\theta r, \partial_{\theta\theta} r) = F_1(r)\partial_{\theta\theta} r + F_2(r)(\partial_\theta r, \partial_\theta r) + G_3(r)\partial_\theta r + G_4(r),$$

with F_1 and F_2 are given respectively by (25) and (26), and where G_3 and G_4 are given respectively by (27) and (28) replacing u by 0.

Since the coefficients of Equation (29) does not depend on θ, if we consider an initial data r_0 constant in θ, then the solution remains constant in the variable θ and satisfies the ordinary differential equation:

$$\frac{dr}{dt} = Ar + G_4(r), \tag{30}$$

where A is the matrix:

$$A = \begin{pmatrix} -\alpha & -1 \\ 1 & -\alpha \end{pmatrix} \begin{pmatrix} -\frac{b}{\lambda(a+b)} & 0 \\ 0 & \frac{a-b}{\lambda(a+b)} - 1 \end{pmatrix},$$

and where $G_4 : B(0, \frac{1}{2}) \longrightarrow \mathbb{R}^2$ satisfies $G_4(r) = \mathscr{O}(|r|^2)$ in a neighbourhood of zero. In order to prove that 0 is unstable for (29), it suffices to prove that zero is unstable for the o.d.e. (30). Writing $A = \dfrac{b}{\lambda(a+b)}B$, where

$$B = \begin{pmatrix} \alpha & -\delta \\ -1 & -\alpha\delta \end{pmatrix}, \text{ with } \delta = \frac{a}{b} - 1 - \frac{\lambda(a+b)}{b},$$

we have to prove that for all $\alpha > 0$ and all $\delta \in \mathbb{R}$, B admits at least one eigenvalue with strictly positive real part. We denote by $\Delta = \alpha^2(1-\delta)^2 + 4(1+\alpha^2)\delta$ the discriminant of the characteristic polynomial $X^2 - \alpha(1-\delta)X - (1+\alpha^2)\delta$.

If $\Delta \le 0$, then $\delta < 0$ and the real part of the eigenvalues equals $\alpha(1-\delta)$, which is positive.

If $\Delta > 0$, the eigenvalues are real. If $\delta > 0$, there product is negative, so that one of them is in $\mathbb{R}^{+*}$. If $\delta = 0$, then $\alpha > 0$ is an eigenvalue of B. If $\delta < 0$, the eigenvalues have the same sign, and their sum equals $\alpha(1-\delta) > 0$, so they are positive.

Therefore, for all $\delta \in \mathbb{R}$, for all $\alpha > 0$, B admits at least one eigenvalue with strictly positive real part. So, the same result occurs for A, and the null solution is unstable for (30). This concludes the proof of Theorem 3.

□

V. STABILITY RESULTS FOR e_θ

The solution e_θ corresponds to $u \equiv \frac{\pi}{2}$, so Equation (6) writes

$$\partial_t r = \begin{pmatrix} -\alpha & -1 \\ 1 & -\alpha \end{pmatrix} \mathscr{L} r + G(r, \partial_\theta r, \partial_{\theta\theta} r), \tag{31}$$

where

$$\mathscr{L} = \begin{pmatrix} -\partial_{\theta\theta} + \frac{b}{\lambda(a+b)} & 0 \\ 0 & -\partial_{\theta\theta} + \frac{a}{\lambda(a+b)} - 1 \end{pmatrix},$$

and

$$G(r, \partial_\theta r, \partial_{\theta\theta} r) = F_1(r)\partial_{\theta\theta} r + F_2(r)(\partial_\theta r, \partial_\theta r) + G_3(r)\partial_\theta r + G_4(r),$$

with F_1 and F_2 are given respectively by (25) and (26), and where G_3 and G_4 are given respectively by (27) and (28) replacing u by $\frac{\pi}{2}$.

A. Instability for $\lambda > \frac{a}{a+b}$

As in the previous section, since the coefficients of (31) do not depend of θ, we focus on constant-in-θ perturbations of 0, and we are led to study the instability of zero for the o.d.e:

$$\frac{dr}{dt} = Ar + G_4(r), \tag{32}$$

where A is the matrix:

$$A = \begin{pmatrix} -\alpha & -1 \\ 1 & -\alpha \end{pmatrix} \begin{pmatrix} \frac{b}{\lambda(a+b)} & 0 \\ 0 & \frac{a}{\lambda(a+b)} - 1 \end{pmatrix},$$

We define $\mu \in \mathbb{R}^{*+}$ by $\mu = -\frac{a-\lambda(a+b)}{b}$, so that $A = \frac{b}{\lambda(a+b)}B$, with $B = \begin{pmatrix} -\alpha & \mu \\ 1 & \alpha\mu \end{pmatrix}$. The characteristic polynomial of B has a strictly positive discriminant, so that the eigenvalues are real. In addition, their product equals $-(1+\alpha^2)\mu < 0$, thus one eigenvalue of B is strictly positive. So A admit a strictly positive eigenvalue, therefore 0 is unstable for (32), which implies that e_θ is unstable for (6).

B. Asymptotic stability for $\lambda < \frac{a}{a+b}$

We consider now the case $\lambda < \frac{a}{a+b}$. In this case, both $\mathscr{L}_1 = -\partial_{\theta\theta} + \frac{b}{\lambda(a+b)}$ and $\mathscr{L}_2 = -\partial_{\theta\theta} + \frac{a}{\lambda(a+b)} - 1$ are self-adjoint coercive operators, so there exists $c_1 > 0$ and $c_2 > 0$ such that for all $r \in H^2_p$ we have:

$$c_1 \|r\|^2_{H^1_p} \le < \mathscr{L} r \,|\, r > \le c_2 \|r\|^2_{H^1_p}, \tag{33}$$

and

$$c_1 \|r\|^2_{H^2_p} \le \|\mathscr{L} r\|^2_{L^2_p} \le c_2 \|r\|^2_{H^2_p}. \tag{34}$$

We have the following estimate concerning the nonlinear part of (20):

Lemma 1. *There exists a constant c such that for all* $r \in H^2_p$ *with* $\|r\|_{L^\infty} \leq \frac{1}{2}$*, we have:*

$$\|F(\theta, r, \partial_\theta r, \partial_{\theta\theta} r)\|_{L^2_p} \leq c < \mathscr{L} r \,|\, r >^{\frac{1}{2}} \|\mathscr{L} r\|_{L^2_p}.$$

Proof. Using Proposition 1, assuming that $\|r\|_{L^\infty} \leq \frac{1}{2}$, we have:

$$\begin{aligned}
\|F_1(r)\|_{L^2_p} &\leq \|F_1(r)\|_{L^\infty} \|\partial_{\theta\theta} r\|_{L^2_p}, \\
&\leq K\|r\|_{L^\infty}\|r\|_{H^2_p}, \\
&\leq K'\|r\|_{H^1_p}\|r\|_{H^2_p} \\
&\quad \text{by Soboblev embeddings,}
\end{aligned}$$

$$\begin{aligned}
\|F_2(r)(\partial_\theta r, \partial_\theta r)\|_{L^2_p} &\leq \|F_2(r)\|_{L^\infty}\|\partial_\theta r\|^2_{L^4}, \\
&\leq K\|r\|_{L^\infty}\|\partial_{\theta\theta} r\|_{L^2_p} \\
&\quad \text{by Gagliardo Nirenberg inequality,} \\
&\leq K'\|r\|_{H^1_p}\|r\|_{H^2_p} \\
&\quad \text{by Soboblev embeddings,}
\end{aligned}$$

$$\begin{aligned}
\|F_3(\cdot, r)\partial_\theta r\|_{L^2} &\leq \|F_3(\cdot, r)\|_{L^\infty}\|\partial_\theta r\|_{L^2}, \\
&\leq K\|r\|_{L^\infty}\|\partial_\theta r\|_{L^2}, \\
&\leq K'\|r\|_{H^1_p}\|r\|_{H^2_p},
\end{aligned}$$

$$\begin{aligned}
\|F_4(r)\|_{L^2} &\leq K\|r\|^2_{L^4}, \\
&\leq K'\|r\|^2_{H^1_p} \\
&\quad \text{by Soboblev embeddings,} \\
&\leq K'\|r\|_{H^1_p}\|r\|_{H^2_p}.
\end{aligned}$$

Adding up these estimates, using (33) and (34), we conclude the proof of Lemma 1.

□

Let us prove that If $\lambda < \frac{a}{a+b}$, e_θ is asymptotically stable.

Proof. Taking the scalar product in L^2_p of the equation (20) with $\mathscr{L} r$, we obtain:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} < \mathscr{L} r \,|\, r > &= -\alpha\|\mathscr{L} r\|^2_{L^2_p} + < F \,|\, \mathscr{L} r >, \\
&\leq -\alpha\|\mathscr{L} r\|^2_{L^2_p} + \|F\|_{L^2_p}\|\mathscr{L} r\|_{L^2_p}.
\end{aligned}$$

So, by Proposition 1, as long as $\|r\|_{L^\infty} \leq \frac{1}{2}$, we have:

$$\frac{1}{2}\frac{d}{dt} < \mathscr{L} r \,|\, r > \leq -\alpha\|\mathscr{L} r\|^2_{L^2_p} + c < \mathscr{L} r \,|\, r >^{\frac{1}{2}} \|Lr\|^2_{L^2_p},$$

that is:

$$\frac{1}{2}\frac{d}{dt} < \mathscr{L} r \,|\, r > + \left(\alpha - c < \mathscr{L} r \,|\, r >^{\frac{1}{2}}\right)\|Lr\|^2_{L^2_p} \leq 0.$$

Using the Sobolev embedding $H^1 \subset L^\infty$ in 1d and Estimate (33), we fix $\eta > 0$ such that $\eta \leq \frac{\alpha}{2c}$ and such that

$$< \mathscr{L} r \,|\, r >^{\frac{1}{2}} \leq \eta \Longrightarrow \|r\|_{L^\infty} \leq \frac{1}{2}.$$

As long as $< Lr, r >^{\frac{1}{2}} \leq \eta$, we obtain $\alpha - c < Lr, r >^{\frac{1}{2}} \geq \frac{\alpha}{2}$ and we have

$$\frac{d}{dt} < \mathscr{L} r \,|\, r > + \alpha\|Lr\|^2_{L^2_p} \leq 0.$$

From Estimates (33) and (34), we have $\|\mathscr{L} r\|^2_{L^2_p} \geq \frac{c_1}{c_2} < \mathscr{L} r \,|\, r >$, therefore, as long as $< \mathscr{L} r \,|\, r >^{\frac{1}{2}} \leq \eta$, we have:

$$\frac{d}{dt} < \mathscr{L} r \,|\, r > + \alpha\frac{c_1}{c_2} < \mathscr{L} r \,|\, r > \leq 0.$$

Using a comparison lemma, we deduce that as long as $< \mathscr{L} r \,|\, r >^{\frac{1}{2}} \leq \eta$:

$$< \mathscr{L} r(t) \,|\, r(t) > \;\leq\; < \mathscr{L} r(0) \,|\, r(0) > e^{-\alpha\frac{c_1}{c_2}t}. \tag{35}$$

We fix $\eta_0 = \frac{\eta}{2}$. Suppose that:

$$< \mathscr{L} r(0) \,|\, r(0) >^{\frac{1}{2}} \leq \eta_0. \tag{36}$$

So, we have:

$$\forall t \geq 0, \quad < \mathscr{L} r(t) \,|\, r(t) >^{\frac{1}{2}} < \eta. \tag{37}$$

If it is not the case, there exists $t' > 0$ such that:

$$< \mathscr{L} r(t') \,|\, r(t') >^{\frac{1}{2}} \geq \eta.$$

Let t_1 be the smallest time in which (37) is not satisfied. Then at time t_1, we have:

$$< \mathscr{L} r(t_1) \,|\, r(t_1) >^{\frac{1}{2}} = \eta. \tag{38}$$

For all $t \in [0, t_1[$, we have

$$< \mathscr{L} r(t) \,|\, r(t) > \;\leq\; < \mathscr{L} r(0) \,|\, r(0) > e^{-\alpha\frac{c_1}{c_2}t} \leq \eta_0^2 = \frac{1}{4}\eta^2.$$

By continuity, we have $< \mathscr{L} r(t_1) \,|\, r(t_1) > \leq \frac{1}{4}\eta^2$ and this is in contradiction with (38).

Finally, under the assumption $< \mathscr{L} r(0) \,|\, r(0) >^{\frac{1}{2}} \leq \eta_0$, for all positive time, Estimate (35) is valid, so that $\|r\|_{H^1}$ is small for all t and $\|r\|_{H^1_p}$ tends to zero as t goes to infinity. □

C. Critical case $\lambda = \frac{a}{a+b}$

In this case, the effective field writes:

$$\mathscr{H}_{\text{eff}}(m) = \frac{\partial^2 m}{\partial\theta^2} - \frac{b}{a}(m \cdot e_r)e_r - m_3 e_3.$$

We remark that there exists a one-parameter family of steady-state solutions for Eq. (6) given by:

$$\mathbf{m}_\varphi(\theta) = \cos\varphi e_\theta(\theta) + \sin\varphi e_3.$$

By projection on the mobile frame (M^1, M^2), we obtain that (20) admits a one-parameter family of steady-state solutions given by:

$$\mathbf{V}(\varphi)(\theta) = \begin{pmatrix} \mathbf{m}_\varphi(\theta) \cdot e_r(\theta) \\ \mathbf{m}_\varphi(\theta) \cdot e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \sin\varphi \end{pmatrix}. \tag{39}$$

The existence of this one-parameter family induces that 0 is in the spectrum of the linearization $\mathscr{L}$, as it can be seen from the fact that:

$$\mathscr{L} = \begin{pmatrix} -\partial_{\theta\theta} + \frac{b}{a} & \\ & -\partial_{\theta\theta} \end{pmatrix},$$

We claim the following Proposition:

Proposition 2. *We assume that there exists a function* $\mathbf{V}$ *of class* $\mathscr{C}^1$*, defined on* $]-\tau_0,\tau_0[$ *(*$\tau_0>0$*), with values in* $\mathscr{C}_p^\infty$*, such that:*

- $\forall \varphi \in]-\tau_0,\tau_0[$, $\mathbf{V}(\varphi)$ *is a steady state for* (20)*,*
- $\mathbf{V}(0)=(0,0)$*,* $\mathbf{V}'(0)\neq 0$*,*
- $Ker\mathscr{L} = Vect\ \mathbf{V}'(0)$*,*

We assume in addition that:

$$\exists c>0, \forall v \in (\ Ker\mathscr{L})^\perp \cap H_p^2, < \mathscr{L}v\,|\,v > \geq c\|v\|_{L_p^2}^2. \quad (40)$$

Then 0 *is stable for* (20)*.*

In addition, there exists $\eta_0>0$ *such that for all solution* r *of* (20) *such that* $\|r(0,\cdot)\|_{H_p^1} \leq \eta_0$*, then there exists* $\varphi_\infty \in]-\tau_0,\tau_0[$ *such that* $\|r(t,\cdot)-\mathbf{V}(\varphi_\infty)(\cdot)\|_{H_p^1}$ *tends to zero when* t *tends to* $+\infty$*.*

Proposition 2 will be established in subsection **E**. We prove in subsection **D.** that e_θ in the critical case falls under this proposition.

D. Application of Proposition 2 in the critical case

Let us check that the assumptions of Proposition 2 are satisfied in the case $\lambda = \frac{a}{a+b}$ for the steady state e_θ, with $\mathbf{V}$ given by (39). Indeed, on the one hand,

$$\forall \theta,\ \mathbf{V}'(0)(\theta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so that Ker $\mathscr{L} = \mathbb{R}\mathbf{V}'(0)$, and on the other hand, for all $v = (v_1,v_2) \in (\ \text{Ker}\mathscr{L})^\perp \cap H_p^2$,

$$< \mathscr{L}v\,|\,v > = \int_0^{2\pi} \left(|\partial_x v_1|^2 + \frac{b}{a}|v_1|^2 \right) + \int_0^{2\pi} |\partial_x v_2|^2$$

$$\geq \frac{b}{a}\|v_1\|_{L_p^2}^2 + \|v_2\|_{L_p^2}^2$$

by Poincaré-Wirtinger inequality which can be applied to v_2, since $v \in (\text{Ker}\ \mathscr{L})^\perp$ implies that $\int_0^{2\pi} v_2 = 0$.

Therefore, by Proposition 2, 0 is stable for (20), and starting from an initial datum close to 0, a solution r for (20) satisfies that there exists φ_∞ such that $r(t,\cdot)$ tends to $\mathbf{V}(\varphi_\infty)$ in H_p^1 when t tends to $+\infty$. This implies that e_θ is stable for (6) and that if m is a solution for (6), if $m(0,\cdot)-e_\theta$ is sufficiently small in H_p^1, then there exists φ_∞ such that $m(t,\cdot)$ tends to $\mathbf{m}_{\varphi_\infty}$ when t tends to $+\infty$.

E. Proof of Proposition 2

We remark that $\mathscr{L}$ is a positive self-adjoint operator with compact resolvent and admits 0 as a simple eigenvalue. We define $L_p^{2,\perp}$ and $H_p^{k,\perp}$ by

$$L_p^{2,\perp} = \left\{ v \in (L_p^2)^2, < v\,|\,\mathbf{V}'(0) >= 0 \right\}$$

$$H_p^{k,\perp} = (H_p^k)^2 \cap L_p^{2,\perp}.$$

From (40), we have the following estimates:

Proposition 3. *There exists* $c_1>0$ *and* $c_2>0$ *such that for all* $w \in H_p^{2,\perp}$ *we have:*

$$c_1\|w\|_{H_p^1}^2 \leq\ < \mathscr{L}w\,|\,w >\ \leq\ c_2\|w\|_{H_p^1}^2,$$

$$c_1\|w\|_{H_p^2}^2 \leq\ \|\mathscr{L}w\|_{L_p^2}^2\ \leq\ c_2\|w\|_{H_p^2}^2.$$

1. New unknown

In order to deal with the null eigenvalue, we rewrite the unknown r in the following system of coordinates:

$$r(t,\theta) = \mathbf{V}(\varphi(t))(\theta) + w(t,\theta), \quad (41)$$

where $\varphi \in \mathscr{C}^1(\mathbb{R}^+;\mathbb{R})$ and $w \in \mathscr{C}^1(\mathbb{R}^+;H_p^2)$ such that w satisfies the orthogonality condition:

$$\forall t>0, \quad < w\,|\,\mathbf{V}'(0) >= 0.$$

The decomposition (41) exists while r remains in a neighbourhood of zero as it is established in the following proposition:

Proposition 4. *There exists* $\eta_0>0$ *such that for all* $\mathbf{r} \in (L_p^2)^2$ *with* $\|\mathbf{r}\|_{L^\infty} \leq \eta_0$*, there exists a unique pair* $(\varphi,w) \in \mathbb{R}\times L_p^{2,\perp}$ *such that*

$$\mathbf{r} = \mathbf{V}(\varphi) + w. \quad (42)$$

In addition, if $\mathbf{r} \in H_p^k$ *then* $w \in H_p^{k,\perp}$*.*

Proof. Suppose that $\mathbf{r}$ admits a decomposition as (42). Taking the L_p^2-inner product of $\mathbf{r}$ with $\mathbf{V}'(0)$, we obtain that

$$< \mathbf{r}\,|\,\mathbf{V}'(0) > = < \mathbf{V}(\varphi)\,|\,\mathbf{V}'(0) > .$$

We define the function ψ by:

$$\begin{array}{rccl} \psi: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & s & \longmapsto & < \mathbf{V}(s)\,|\,\mathbf{V}'(0) > \end{array}$$

since $\psi(0)=0$ and $\psi'(0) = \|\mathbf{V}'(0)\|_{L_p^2}^2 > 0$ then ψ is a $\mathscr{C}^1$-diffeomorphism in a neighborhood of zero, so φ is characterized by:

$$\varphi = \psi^{-1}\left(< \mathbf{r}\,|\,\mathbf{V}'(0) > \right).$$

By subtraction we obtain that w is characterized by $w = \mathbf{r} - \mathbf{V}(\varphi)$ which is in $L_p^{2,\perp}$. □

Plugging (41) in (20), we find the following equation equivalent to (20):

$$\frac{d\varphi}{dt}\,\mathbf{V}'(\varphi) + \partial_t w = \Lambda w + G(\theta,\varphi,w,\partial_\theta w,\partial_{\theta\theta} w), \qquad (43)$$

where

$$G = G_1 + G_2 + G_3 + G_4,$$

with

$$\begin{aligned} G_1 &= F_1(r)(\partial_{\theta\theta} w) + \tilde{F}_1(r)(w)(\partial_{\theta\theta}\mathbf{V}(\varphi)),\\ G_2 &= 2F_2(r)(\partial_\theta w, \partial_\theta \mathbf{V}(\varphi)) + F_2(r)(\partial_\theta w, \partial_\theta w)\\ &\quad + \tilde{F}_2(r)(w)(\partial_\theta \mathbf{V}(\varphi), \partial_\theta \mathbf{V}(\varphi)), \qquad (44)\\ G_3 &= F_3(\theta,r)(\partial_\theta w) + \tilde{F}_3(\theta,r)(w)(\partial_\theta \mathbf{V}(\varphi)),\\ G_4 &= \tilde{F}_4(r)(w). \end{aligned}$$

In (44), we denote $r = \mathbf{V}(\varphi) + w$, and the terms $\tilde{F}_i(r)$ are obtained by writing the Taylor expansion of F_i at the point $\mathbf{V}(\varphi)$:

$$\tilde{F}_i(r) = F_i(r) - F_i(\mathbf{V}(\varphi)) = \int_0^1 \mathrm{d}F_i(\mathbf{V}(\varphi) + sw)ds.$$

We remark that $\tilde{F}_1(r)$ and $\tilde{F}_3(r)$ are in $\mathscr{L}(\mathbb{R}^2;\mathscr{L}(\mathbb{R}^2;\mathbb{R}^2))$, $\tilde{F}_2(r) \in \mathscr{L}(\mathbb{R}^2;\mathscr{L}_2(\mathbb{R}^2\times\mathbb{R}^2;\mathbb{R}^2))$, and $\tilde{F}_4(r) \in \mathscr{L}(\mathbb{R}^2;\mathbb{R}^2)$.

Taking the L_p^2-inner product of (43) with $\mathbf{V}'(0)$, we obtain:

$$\frac{d\varphi}{dt} g(\varphi) = <\Lambda w\,|\,\mathbf{V}'(0)> + <G\,|\,\mathbf{V}'(0)>, \qquad (45)$$

where

$$g(\varphi) = <\mathbf{V}'(\varphi)\,|\,\mathbf{V}'(0)> .$$

Since $g(0) = \|\mathbf{V}'(0)\|^2_{L^2_p} \neq 0$, there exists $\eta_1 > 0$ such that if $|\varphi| \le \eta_1$, then $|g(\varphi)| \ge \frac{1}{2}|g(0)|$. So we can divide Eq. (45) by $g(\varphi)$ and we obtain the following equation, valid for $|\varphi| \le \eta_1$ and for w such that $\|\mathbf{V}(\varphi)+w\|_{L^\infty_p} < 1$:

$$\frac{d\varphi}{dt} = \mathscr{H}(\varphi,w) =: \frac{1}{g(\varphi)}\left(<\Lambda w\,|\,\mathbf{V}'(0)> + <G\,|\,\mathbf{V}'(0)>\right). \qquad (46)$$

Replacing $\frac{d\varphi}{dt}$ in equation (43) by the expression given in (46) we find:

$$\partial_t w = \Lambda w + \mathscr{F}(\theta,\varphi,w,\partial_\theta w,\partial_{\theta\theta} w), \qquad (47)$$

where $\mathscr{F} = G - \mathscr{H}(\varphi,w)\partial_s \mathbf{V}(\varphi)$.

2. *Estimate of the nonlinear part*

Using Sobolev embedding of H^1 into L^∞, using Proposition 3, we introduce η_2, $0 < \eta_2 \le \eta_1$ such that if $|\varphi| \le \eta_2$ and $< Lw\,|\,w >^{\frac{1}{2}} \le \eta_2$, then $\|\mathbf{V}(\varphi)+w\|_{L^\infty_p} \le \frac{1}{2}$.

Proposition 5. *There exists a constant $C > 0$ such that as long as $<\mathscr{L}w\,|\,w>^{\frac{1}{2}} \le \eta_2$ and $|\varphi| \le \eta_2$, then*

$$|\mathscr{H}(\varphi,w)| \le C <\mathscr{L}w\,|\,w>^{\frac{1}{2}},$$

and

$$\|\mathscr{F}(\theta,\varphi,w,\partial_\theta w,\partial_{\theta\theta} w)\|_{L^2_p} \le C\left(|\varphi| + <\mathscr{L}w\,|\,w>^{\frac{1}{2}}\right)\|\mathscr{L}w\|_{L^2_p}$$

Proof. Using $\mathbf{V}$ is in $\mathscr{C}^1(]-\tau_0,\tau_0[;\mathscr{C}^\infty_p)$ with $\mathbf{V}(0) = 0$, there exists a constant K such that for s in a neighbourhood of 0 and for all $\theta \in \mathbb{R}$, we have:

$$|\mathbf{V}(s)(\theta)| \le K|s|, \quad |\partial_\theta \mathbf{V}(s)(\theta)| \le K, \quad |\partial_{\theta\theta}\mathbf{V}(s)(\theta)| \le K.$$

Using Proposition 1, we estimate each term of G on the following way: starting by G_1, we have:

$$\begin{aligned} |G_1| &= |F_1(r)(\partial_{\theta\theta} w) + \tilde{F}_1(r)(w)(\partial_{\theta\theta}\mathbf{V}(\varphi))|\\ &\le C\left(|r||\partial_{\theta\theta} w| + |r||w|\right), \end{aligned}$$

then

$$\begin{aligned} \|G_1\|_{L^2_p} &\le C\|\mathbf{V}(\varphi)+w\|_{L^\infty_p}\left(\|\partial_{\theta\theta} w\|_{L^2_p} + \|w\|_{L^2_p}\right)\\ &\le C\left(|\varphi| + \|w\|_{L^\infty_p}\right)\|w\|_{H^2_p}. \end{aligned}$$

It comes that

$$\|G_1\|_{L^2_p} \le C\left(|\varphi| + \|w\|_{H^1_p}\right)\|w\|_{H^2_p}.$$

In addition, on the one hand we have:

$$\begin{aligned} <F_1(r)(\partial_{\theta\theta} w)\,|\,\mathbf{V}'(0)> &= <\partial_{\theta\theta} w\,|\,(F_1(r))^*(\mathbf{V}'(0))>\\ &= -<\partial_\theta w\,|\,\partial_\theta((F_1(r))^*(\mathbf{V}'(0)))> \end{aligned}$$

so

$$\left|<F_1(r)(\partial_{\theta\theta} w)\,|\,\mathbf{V}'(0)>\right| \le C\|w\|_{H^1_p} \le C <\mathscr{L}w\,|\,w>^{\frac{1}{2}} .$$

On the other hand,

$$\begin{aligned} &|<\tilde{F}_1(r)(w)(\partial_{\theta\theta}\mathbf{V}(\varphi))\,|\,\mathbf{V}'(0)>| \le\\ &\qquad \|\tilde{F}_1(r)\|_{L^\infty_p}\|w\|_{L^\infty_p}\|\mathbf{V}(\varphi)\|_{H^2_p}\|\mathbf{V}'(0)\|_{L^1_p}\\ &\qquad \le C\|w\|_{H^1_p} \le C <\mathscr{L}w\,|\,w>^{\frac{1}{2}} . \end{aligned}$$

Therefore, there exists a constant C such that

$$|<G_1\,|\,\mathbf{V}'(0)> \le C <\mathscr{L}w\,|\,w>^{\frac{1}{2}} .$$

Concerning G_2, we have:

$$\begin{aligned} |G_2| \le\ & |2F_2(r)(\partial_\theta w, \partial_\theta\mathbf{V}(\varphi)) + F_2(r)(\partial_\theta w, \partial_\theta w)|\\ & + |\tilde{F}_2(r)(w)(\partial_\theta\mathbf{V}(\varphi), \partial_\theta\mathbf{V}(\varphi))|,\\ \le\ & C|r||\partial_\theta w||\partial_\theta\mathbf{V}(\varphi)| + C|r||\partial_\theta w|^2 + C|w||\partial_\theta\mathbf{V}|^2,\\ \le\ & C\left(|\varphi| + |w|\right)\left(|\partial_\theta w| + |\partial_\theta w|^2\right) + C|w||\partial_\theta\mathbf{V}|^2. \end{aligned}$$

Then

$$\|G_2\|_{L^2_p} \leq \; C\left(|\varphi| + \|w\|_{L^\infty_p}\right)\left(\|\partial_\theta w\|_{L^2_p} + \|\partial_\theta w\|_{L^4_p}\right)$$
$$+C\|w\|_{L^2_p}\|\partial_\theta \mathbf{V}\|_{L^4_p}.$$

Using that $\|\partial_\theta v\|_{L^4_p} \leq \|v\|_{L^\infty_p}\|\partial_{\theta\theta} v\|_{L^2_p}$, we obtain:

$$\|G_2\|_{L^2_p} \leq \; C\left(\left(|\varphi| + \|w\|_{L^\infty_p}\right)\|w\|_{H^2_p} + \|w\|_{H^2_p}|\varphi|\right)$$
$$\leq \; C\left(|\varphi| + \|w\|_{H^1_p}\right)\|w\|_{H^2_p}.$$

We have also that:

$$|< G_2\,|\,\mathbf{V}'(0) >| \leq \|G_2\|_{L^1_p}\|\mathbf{V}'(0)\|_{L^\infty_p},$$

and since

$$\|G_2\|_{L^1_p} \leq C(|\varphi| + \|w\|_{L^\infty_p})(\|\partial_\theta\|_{L^1_p} + \|\partial_\theta\|^2_{L^2}) + C\|w\|_{L^2_p}\|\partial_\theta \mathbf{V}\|^2_{L^4_p},$$

we obtain that

$$|< G_2\,|\,\mathbf{V}'(0) >| \leq C\|w\|_{H^1_p}.$$

Concerning G_3, we have:

$$|G_3| = \; |F_3(\theta,r)(\partial_\theta w) + \tilde{F}_3(\theta,r)(w)(\partial_\theta \mathbf{V}(\varphi))|$$
$$\leq \; C|r||\partial_\theta w| + K|w||\partial_\theta \mathbf{V}_\varphi|.$$

Thus

$$\|G_3\|_{L^2_p} \leq C\left(|\varphi| + \|w\|_{L^\infty_p}\right)\|\partial_\theta w\|_{L^2_p} + K\|w\|_{L^2_p}|\varphi|.$$

So,

$$|< G_3\,|\,\mathbf{V}'(0) >| \leq C\|w\|_{H^1_p} \leq C < \mathscr{L}w\,|\,w >^{\frac{1}{2}},$$

and

$$\|G_3\|_{L^2_p} \leq C\left(|\varphi| + \|w\|_{H^1_p}\right)\|w\|_{H^2_p}.$$

We finally estimate G_4:

$$\|G_4\|_{L^2_p} = \|\tilde{F}_4(r)(w)\|_{L^2_p} \leq C\left(|\varphi| + \|w\|_{L^\infty_p}\right)\|w\|_{L^2_p}.$$

So, we obtain both that:

$$\|G_4\|_{L^2_p} \leq C\left(|\varphi| + \|w\|_{H^1_p}\right)\|w\|_{H^2_p},$$

and that:

$$|< G_4\,|\,\mathbf{V}'(0) >| \leq C < \mathscr{L}w\,|\,w >^{\frac{1}{2}}.$$

Adding up all the previous estimates, we obtain first that:

$$\|G\|_{L^2_p} \leq C\left(|\varphi| + \|w\|_{H^1_p}\right)\|w\|_{H^2_p}.$$

Using Proposition 3 we get

$$\|G\|_{L^2_p} \leq C\left(|\varphi| + < \mathscr{L}w\,|\,w >^{\frac{1}{2}}\right)\|\mathscr{L}w\|_{L^2_p}. \quad (48)$$

In addition, we remark that:

$$\begin{aligned} < \Lambda w\,|\,\mathbf{V}'(0) > &= \; < \mathscr{L}w\,|\begin{pmatrix} -\alpha & 1 \\ -1 & -\alpha \end{pmatrix}\mathbf{V}'(0) > \\ &= \; < w\,|\,\mathscr{L}\begin{pmatrix} -\alpha & 1 \\ -1 & -\alpha \end{pmatrix}\mathbf{V}'(0) > \\ &= \; < w\,|\,\mathscr{L}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{V}'(0) > . \end{aligned}$$

since $\mathscr{L}$ is self-adjoint and since $\mathscr{L}\mathbf{V}'(0) = 0$. Therefore,

$$|< \Lambda w\,|\,\mathbf{V}'(0) >| \leq K\|w\|_{L^2_p}.$$

Using the previous estimates, we obtain that:

$$\left|< \Lambda w\,|\,\mathbf{V}'(0) > + < G\,|\,\mathbf{V}'(0) >\right| \leq C < \mathscr{L}w\,|\,w >^{\frac{1}{2}}. \quad (49)$$

Next, we recall that since $|\varphi| \leq \eta_1$, $|g(\varphi)|$ is bounded by below by $\frac{1}{2}g(0)|$. So we obtain that:

$$|\frac{1}{g(\varphi)}| \leq \frac{2}{g(0)}. \quad (50)$$

Coupling (49) and (50), we obtain that there exists C such that

$$|\mathscr{H}(\varphi,w)| \leq C < \mathscr{L}w\,|\,w >^{\frac{1}{2}}, \quad (51)$$

and coupling (48) and (51), we conclude the proof of Proposition 5.

□

3. *Proof of the stability*

Taking the scalar product in L^2_p of (47) with Lw we find:

$$\frac{1}{2}\frac{d}{dt} < \mathscr{L}w\,|\,w > +\alpha\|Lw\|^2_{L^2_p} = < \mathscr{F}\,|\,w > . \quad (52)$$

Using Proposition 5, as long as $< Lw\,|\,w >^{\frac{1}{2}} \leq \eta_0$ and $|\varphi| \leq \eta_0$, we have:

$$\frac{1}{2}\frac{d}{dt} < \mathscr{L}w\,|\,w > + \left(\alpha - C\left(|\varphi| + < \mathscr{L}w\,|\,w >^{\frac{1}{2}}\right)\right)\|Lw\|^2_{L^2_p} \leq 0.$$

As long as $< \mathscr{L}w\,|\,w >^{\frac{1}{2}} \leq \eta_1$ and $|\varphi| \leq \min\{\eta_0, \frac{\alpha}{2C}\}$ we find

$$\frac{1}{2}\frac{d}{dt} < \mathscr{L}w\,|\,w > + \left(\frac{\alpha}{2} - C < \mathscr{L}w\,|\,w >^{\frac{1}{2}}\right)\|\mathscr{L}w\|^2_{L^2_p} \leq 0.$$

Let $\eta_2 = \min\{\eta_1, \frac{\alpha}{4C}\}$. As long as $< \mathscr{L}w\,|\,w >^{\frac{1}{2}} \leq \eta_2$, we have:

$$\frac{d}{dt} < \mathscr{L}w\,|\,w > + \frac{\alpha}{2}\|\mathscr{L}w\|^2_{L^{2,\perp}_p} \leq 0.$$

Using comparison lemma, we deduce that as long as $< \mathscr{L}w\,|\,w >^{\frac{1}{2}} \leq \eta_2$, we have:

$$< \mathscr{L}w(t)\,|\,w(t) > \;\leq\; < \mathscr{L}w(0)\,|\,w(0) > e^{\frac{-\alpha k}{2}t}. \qquad (53)$$

As in the proof of the stability of e_θ in Section V B, we can prove that if $< \mathscr{L}w(0)\,|\,w(0) >^{\frac{1}{2}} \leq \frac{\eta_2}{2}$ then $\|w\|_{H^1}$ is small for all time t and it tends to zero as t goes to infinity.

In addition, using Estimate (53) together with Equation (46) and the estimate on $\mathscr{H}$ in Proposition 5, we obtain that:

$$\forall t \geq 0, \quad \left|\frac{d\varphi}{dt}\right| \leq C < \mathscr{L}w(0)\,|\,w(0) > e^{\frac{-\alpha k}{2}t},$$

and by integration, we obtain that if $\varphi(0)$ and $< \mathscr{L}w(0)\,|\,w(0) >$ are small enough, then for all t, $\varphi(t)$ remains small and tends to a finite limit when t tends to $+\infty$.

This concludes the proof of Proposition 2.

VI. LINEAR INSTABILITY RESULTS

A. Linear instability criterion

Proposition 6. *Let $M^0 = (\cos u)e_r + (\sin u)e_\theta$ be a steady state for Equation* (6). *We assume that $\mathscr{L}$ given by* (21)-(22)-(23) *admits a negative eigenvalue μ associated to the eigenvector ξ. Then the linearised system $\partial_t v = J\mathscr{L}v$ admits solutions $v \in \mathscr{C}^0(\mathbb{R}^+; H^2_p)$ which are unbounded in $L^\infty(\mathbb{R}^+; H^1_p)$.*

Proof. The operator $\mathscr{L}$ is self-adjoint with compact resolvent, so that we can split L^2_p as:

$$L^2_p = E^- \oplus^\perp E^0 \oplus^\perp E^+, \qquad (54)$$

where $E^- = \bigoplus_{\lambda \in \mathrm{sp}(\mathscr{L}) \cap \mathbb{R}^{*-}} \ker(\mathscr{L} - \lambda Id)$, $E^0 = \ker \mathscr{L}$, and E^+ is the closure of $\bigoplus_{\lambda \in \mathrm{sp}(\mathscr{L}) \cap \mathbb{R}^{*+}} \ker(\mathscr{L} - \lambda Id)$ in L^2_p. We remark that both E^- and E^0 are finite-dimensional. In addition, $\mathscr{L}$ is coercive on E^+, thus there exist $c_1 > 0$ and $c_2 > 0$ such that:

$$\forall w \in E^+, \;\; c_1 \|w\|^2_{H^1_p} \leq < \mathscr{L}w\,|\,w > \leq c_2 \|w\|^2_{H^1_p}.$$

We assume that $\mathscr{L}$ admits a strictly negative eigenvalue λ associated to the eigenvector $\xi \in H^2_p$.

We consider the solution $v \in \mathscr{C}^0(\mathbb{R}^+; H^2_p)$ of the Cauchy problem:

$$\begin{cases} \partial_t v = J\mathscr{L}v, \\ v(t=0,\cdot) = \xi(\cdot), \end{cases} \qquad (55)$$

Let us assume that v is bounded in $L^\infty(\mathbb{R}^+; H^1_p)$, *i.e.*

$$\exists C,\, \forall t \geq 0,\; \|v(t)\|_{H^1_p} \leq C. \qquad (56)$$

By taking the L^2_p-inner product of (55) with $\mathscr{L}v$, we obtain that:

$$\frac{1}{2}\frac{d}{dt} < \mathscr{L}v\,|\,v > + \alpha \|\mathscr{L}v\|^2_{L^2_p} = 0, \qquad (57)$$

so that $t \mapsto < \mathscr{L}v(t)\,|\,v(t) >$ is non increasing. In particular, since $< \mathscr{L}v(0)\,|\,v(0) > = < \mathscr{L}\xi\,|\,\xi > = \lambda \|\xi\|^2_{L^2 p} < 0$, and since $< \mathscr{L}v(t)\,|\,v(t) >$ is bounded by below by (56), we obtain that:

$$\lim_{t \to +\infty} < \mathscr{L}v(t)\,|\,v(t) > \in \mathbb{R}^{*-}. \qquad (58)$$

In addition, integrating (57) between 0 and t and using (56), we obtain that for all $t \geq 0$,

$$\int_0^t \|\mathscr{L}v\|^2_{L^2_p} dt \leq 2C,$$

so

$$\mathscr{L}v \in L^2(\mathbb{R}^+; L^2_p), \qquad (59)$$

and by (55),

$$\frac{\partial v}{\partial t} \in L^2(\mathbb{R}^+; L^2_p). \qquad (60)$$

Using (56), we consider a sequence of times t_n tending to $+\infty$ such that $v_n := v(t_n, \cdot)$ converges to $\mathbf{v}$ weakly in H^1_p and strongly in L^2_p. Using (54), we split v_n as $v_n = v_n^- + v_n^0 + v_n^+$, and we have:

$$< \mathscr{L}v_n\,|\,v_n > = < \mathscr{L}v_n^-\,|\,v_n^- > + < \mathscr{L}v_n^+\,|\,v_n^+ > .$$

By projection into each subspace, (v_n^+) tends to $\mathbf{v}^+$ weakly in H^1_p and (v_n^-) tends weakly to $\mathbf{v}^-$ in H^1_p.

Since E^- is finite dimensional, (v_n^-) tends to $\mathbf{v}^-$ strongly for all norms. In particular, $< \mathscr{L}v_n^-\,|\,v_n^- >$ tends to $< \mathscr{L}\mathbf{v}^-\,|\,\mathbf{v}^- >$.

By convexity of $w \mapsto < \mathscr{L}w\,|\,w >$ in $H^1_p \cap E^+$, we obtain that $< \mathscr{L}\mathbf{v}^+\,|\,\mathbf{v}^+ > \leq \underline{\lim} < \mathscr{L}v_n^+\,|\,v_n^+ >$.

Therefore, by summing a limit plus a limit inf, we obtain that:

$$< \mathscr{L}\mathbf{v}\,|\,\mathbf{v} > \leq \underline{\lim} < \mathscr{L}v_n\,|\,v_n > . \qquad (61)$$

We define w_n by $w_n(\cdot) = \int_0^1 v(t_n + s, \cdot)ds$. We have:

$$\begin{aligned} \|w_n - v_n\|^2_{L^2_p} = & \int_{x=0}^{2\pi} \left| \int_{s=0}^{1} (v(t_n+s,x) - v(t_n,x))ds \right|^2 dx, \\ \leq & \int_{x=0}^{2\pi}\int_{s=0}^{1} |v(t_n+s,x) - v(t_n,x)|^2 ds dx, \\ \leq & \int_{x=0}^{2\pi}\int_{s=0}^{1} \left| \int_0^s \frac{\partial v}{\partial t}(t_n+\tau,x)d\tau \right|^2 ds dx, \\ \leq & \int_{x=0}^{2\pi}\int_{\tau=0}^{1} \left| \frac{\partial v}{\partial t}(t_n+\tau,x) \right|^2 d\tau dx. \end{aligned}$$

So,

$$\|w_n - v_n\|^2_{L^2_p} \leq \int_{t_n}^{+\infty} \|\frac{\partial v}{\partial t}(s,\cdot)\|^2_{L^2_p}.$$

Since $\frac{\partial v}{\partial t} \in L^2(\mathbb{R}^+; L^2_p)$, we obtain that $\|w_n - v_n\|_{L^2_p}$ tends to zero so that w_n tends to $\mathbf{v}$ strongly in L^2_p.

Now, we remark that

$$\begin{aligned}\|\mathscr{L} w_n\|^2_{L^2_p} = & \int_{x=0}^{2\pi} \Big| \int_{s=0}^{1} \mathscr{L} v(t_n+s) ds \Big|^2 dx, \\ \leq & \int_{x=0}^{2\pi} \int_{s=0}^{1} |\mathscr{L} v(t_n+s)|^2 ds dx \\ \leq & \int_{t_n}^{+\infty} \|\mathscr{L} v(s)\|^2_{L^2_p} ds.\end{aligned}$$

Using (59), we obtain that $\mathscr{L} w_n$ tends to zero strongly in L^2_p. Since $\mathscr{L} w_n$ tends to $\mathscr{L}\mathbf{v}$ in the distributions, we obtain that $\mathscr{L}\mathbf{v} = 0$. In particular, we have: $< \mathscr{L}\mathbf{v} \,|\, \mathbf{v} >= 0$, which contradicts (58) and (61).

□

Remark 1. *Using spectral theory arguments, it is possible to prove directly that if $\mathscr{L}$ admits a negative eigenvalue, then Λ admits a positive eigenvalue, so that 0 is unstable for (55). The main arguments of this proof, due to J. F. Bony and V. Bruneau, from IMB, UMR CNRS 5251, Université de Bordeaux ([5]) are that for large α, Λ is a small perturbation of $-\alpha\mathscr{L}$, so it admits a positive eigenvalue, and that the imaginary axe is an impassable barrier for $\alpha > 0$.*

B. Proof of Theorem 6

Let $a > 0$, $b > 0$, and $\lambda > 0$ such that $\frac{b}{\lambda(a+b)} > 1$. We consider a steady-state solution $\mathscr{M}^0_j$ for Equation (6), obtained in Theorem 1. We recall that in this case, u is a solution of the pendulum equation inside the separatrix, so that $\rho^2 < \frac{b}{\lambda(a+b)}$.

We remark that $\mathscr{L}_1 \cos u = (\rho^2 - \frac{b}{\lambda(a+b)}) \cos u$. So $\rho^2 - \frac{b}{\lambda(a+b)}$ is a negative eigenvalue corresponding to the eigenvector $\cos u$. Therefore, using Proposition 6, we obtain that zero is linearly unstable for Equation (20), so that the solutions $\mathscr{M}^0_j$ exhibited in Theorem 1 are linearly unstable for (6).

VII. STABILITY RESULTS FOR $k \neq 0$

We fix a and b such that $a > b > 0$. For $\lambda > 0$ and $k \in \mathbb{Z}^*$, we consider the solution $M^0_{k,\lambda}$ obtained in Theorem 2 on the form:

$$M^0_{k,\lambda} = \Big(\cos u^{k,\lambda}(\theta)\Big) e_r(\theta) + \Big(\sin u^{k,\lambda}(\theta)\Big) e_\theta(\theta),$$

where $u^{k,\lambda}$ satisfies:

$$\begin{cases} \dfrac{d^2 u^{k,\lambda}}{d\theta^2} + \dfrac{b}{\lambda(a+b)} \sin u^{k,\lambda} \cos u^{k,\lambda} = 0, \\ u^{k,\lambda}(0) = 0, \quad u^{k,\lambda}(2\pi) = 2k\pi, \; k \neq 0. \end{cases}$$

We denote $\rho^{k,\lambda} = \frac{du^{k,\lambda}}{d\theta}(0)$ and we recall that all $\theta \in \mathbb{R}$,

$$(\rho^{k,\lambda})^2 = \left(\frac{du^{k,\lambda}}{d\theta}(\theta)\right)^2 + \frac{b}{\lambda(a+b)} (\sin u^{k,\lambda}(\theta))^2. \quad (62)$$

We define the self-adjoint operators $\mathscr{L}^{k,\lambda}_1$ and $\mathscr{L}^{k,\lambda}_2$ by:

$$\mathscr{L}^{k,\lambda}_1 = -\partial_{\theta\theta} + \tfrac{b}{\lambda(a+b)} (\sin^2 u^{k,\lambda} - \cos^2 u^{k,\lambda}),$$

$$\mathscr{L}^{k,\lambda}_2 = \mathscr{L}^{k,\lambda}_1 + P^{k,\lambda},$$

where $P^{k,\lambda}(\theta) = \dfrac{a}{\lambda(a+b)} - \left((\rho^{k,\lambda})^2 + 2\dfrac{du^{k,\lambda}}{d\theta}(\theta) + 1\right)$.

By invariance of Equation (6) by rotation-translation, since $u^{k,\lambda}$ is not constant, there exists a one-parameter family $\mathbf{M}_\varphi$ of static solutions for (6) given by:

$$\mathbf{M}_\varphi(\theta) = R(\varphi) M^0_{k,\lambda}(\theta - \varphi),$$

where $R(\varphi)$ is defined by (8). By projection of the mobile frame $(M^1_{k,\lambda}(\theta), M^2)$, we construct a one-parameter family $\mathbf{V}(\varphi)$ of static solutions for (20) given by:

$$\mathbf{V}(\varphi)(\theta) = \begin{pmatrix} \rho^{k,\lambda}(\varphi)(\theta) \\ 0 \end{pmatrix}, \quad (63)$$

where

$$\begin{aligned}\rho^{k,\lambda}(\varphi)(\theta) = & \; \mathbf{M}_\varphi(\theta) \cdot M^1_{k,\lambda}(\theta), \\ = & -\sin u^{k,\lambda}(\theta) \cos u^{k,\lambda}(\theta - \varphi) \\ & + \cos u^{k,\lambda}(\theta) \sin u^{k,\lambda}(\theta - \varphi).\end{aligned}$$

The existence of this one-parameter family induces that 0 is in the spectrum of $\mathscr{L}^{k,\lambda}$, associated to the eigenvector $\partial_\varphi \mathbf{V}(0) = \begin{pmatrix} \frac{du^{k,\lambda}}{d\theta} \\ 0 \end{pmatrix}$. Indeed, we have the following proposition:

Proposition 7. *The operator $\mathscr{L}^{k,\lambda}_1$ is self-adjoint and positive. In addition* $\ker \mathscr{L}^{k,\lambda}_1 = \mathbb{R} \frac{du^{k,\lambda}}{d\theta}$.

Proof. We first remark that in the considered case, $\frac{du^{k,\lambda}}{d\theta}$ never vanishes. We set

$$\ell = \partial_\theta + \frac{b}{\lambda(a+b)} \frac{\sin u^{k,\lambda} \cos u^{k,\lambda}}{\frac{du^{k,\lambda}}{d\theta}}.$$

Then we have $\ell^* \circ \ell = \mathscr{L}^{k,\lambda}_1$. So $\mathscr{L}^{k,\lambda}_1$ is a positive operator.

We have also

$$\mathscr{L}^{k,\lambda}_1 \frac{du^{k,\lambda}}{d\theta} = \left(\frac{d^2 u^{k,\lambda}}{d\theta^2} + \frac{b}{\lambda(a+b)} \cos u^{k,\lambda} \sin u^{k,\lambda}\right)' = 0,$$

then $\frac{du^{k,\lambda}}{d\theta} \in \ker \mathscr{L}^{k,\lambda}_1$.

In addition, since $< \mathscr{L}^{k,\lambda}_1 w \,|\, w >= \|\ell w\|^2$, if $\mathscr{L}^{k,\lambda}_1 w = 0$, then $\ell w = 0$, so $w \in \ker \ell$ which is of dimensional at most one since the operator ℓ is of order one. □

In order to establish the stability of $M^0_{k,\lambda}$, the crucial point is the positivity of $\mathscr{L}^{k,\lambda}_2$. On the one hand, using Proposition 6, we have the following instability results:

Proposition 8. *If $\mathscr{L}_2^{k,\lambda}$ admits negative eigenvalues, then* 0 *is linearly unstable for* (20)*.*

Corollary 1. *We assume that for all $\theta \in \mathbb{R}$, $P^{k,\lambda}(\theta) < 0$. Then* 0 *is linearly unstable for* (20)*.*

Proof. We have

$$< \mathscr{L}_2^{k,\lambda} \frac{du^{k,\lambda}}{d\theta} \mid \frac{du^{k,\lambda}}{d\theta} >= \int_0^{2\pi} P^{k,\lambda} \left| \frac{du^{k,\lambda}}{d\theta} \right|^2 d\theta < 0,$$

so by min max argument, since $\mathscr{L}_2^{k,\lambda}$ is self-adjoint, it admits negative eigenvalues. Therefore, by Proposition 8, 0 is linearly unstable for (20). □

On the other hand, using Proposition 2, we obtain the following results of asymptotic stability modulo translation under positivity assumption for $\mathscr{L}_2^{k,\lambda}$:

Proposition 9. *We assume that $\mathscr{L}_2^{k,\lambda}$ is positive definite. Then $M^0_{k,\lambda}$ is asymptotically stable modulo rotation-translation for Equation* (6)*.*

Proof. We use Proposition 2 with $\varphi \mapsto \mathbf{V}(\varphi)$ given by (63) Then,

$$\partial_\varphi \mathbf{V}(0)(\theta) = \begin{pmatrix} \frac{du^{k,\lambda}}{d\theta}(\theta) \\ 0 \end{pmatrix} \neq 0.$$

In addition, if $\mathscr{L}(v_1, v_2) = 0$, then $\mathscr{L}_1^{k\lambda} v_1 = \mathscr{L}_2^{k,\lambda} v_2 = 0$. Since $\mathscr{L}_2^{k,\lambda}$ is definite positive, then $v_2 = 0$, and since $\ker \mathscr{L}_1^{k,\lambda} = \mathbb{R}\frac{du^{k,\lambda}}{d\theta}$, then:

$$\ker \mathscr{L} = \mathbb{R}\partial_\varphi \mathbf{V}(0).$$

So the assumptions of Proposition 2 are satisfied, thus we conclude first that 0 is stable for (20), so $M_0^{k,\lambda}$ is stable for (6).

In addition, as it is stated in Proposition 2, for initial data close to zero, $r(t)$ tends to $\mathbf{V}(\varphi_\infty)$, which implies that the corresponding flow for (6) $M(t)$ tends to $\mathbf{M}(\varphi_\infty)$ when t tends to $+\infty$.

□

Corollary 2. *We assume that for all $\theta \in \mathbb{R}$, $P^{k,\lambda}(\theta) > 0$. Then $M^0_{k,\lambda}$ is asymptotically stable modulo rotation-translation for Equation* (6)*.*

Proof. Since $\mathscr{L}_2^{k,\lambda} = \mathscr{L}_1^{k,\lambda} + P^{k,\lambda}$, since $\mathscr{L}_1^{k,\lambda}$ is positive by Proposition 7, the assumption of Corollary 2 implies that $\mathscr{L}_2^{k,\lambda}$ is definite positive, and using Proposition 9, we conclude the proof of the corollary. □

A. Estimates on $P^{k,\lambda}$

Case $k \geq 1$. In this case, $\rho^{k,\lambda} > \sqrt{\frac{b}{\lambda(a+b)}}$ and $\frac{du^{k,\lambda}}{d\theta}$ remains positive so that, using (62), we have:

$$\frac{du^{k,\lambda}}{d\theta} = \sqrt{(\rho^{k,\lambda})^2 - \frac{b}{\lambda(a+b)}\sin^2 u^{k,\lambda}}.$$

So, for all $\theta \in \mathbb{R}$,

$$\sqrt{(\rho^{k,\lambda})^2 - \frac{b}{\lambda(a+b)}} \leq \frac{du^{k,\lambda}}{d\theta}(\theta) \leq \rho^{k,\lambda}, \tag{64}$$

and by integrating for $\theta \in [0, 2\pi]$, since $u^{k,\lambda}(2\pi) = u^{k,\lambda}(0) + 2k\pi$, we obtain:

$$\sqrt{(\rho^{k,\lambda})^2 - \frac{b}{\lambda(a+b)}} \leq k \leq \rho^{k,\lambda}. \tag{65}$$

With this estimate, we obtain also that:

$$k \leq \rho^{k,\lambda} \leq \sqrt{k^2 + \frac{b}{\lambda(a+b)}}. \tag{66}$$

Using (64) and (66), on the one hand, we obtain that

$$\begin{aligned}(\rho^{k,\lambda})^2 + 2\frac{du^{k,\lambda}}{d\theta}(\theta) + 1 \leq &\ (\rho^{k,\lambda} + 1)^2, \\ \leq &\ \left(1 + \sqrt{k^2 + \frac{b}{\lambda(a+b)}}\right)^2,\end{aligned}$$

so that, for all θ,

$$P^{k,\lambda}(\theta) \geq \frac{a}{\lambda(a+b)} - \left(1 + \sqrt{k^2 + \frac{b}{\lambda(a+b)}}\right)^2. \tag{67}$$

On the other hand, using (64), we have:

$$\begin{aligned}(\rho^{k,\lambda})^2 + 2\frac{du^{k,\lambda}}{d\theta}(\theta) + 1 \geq &\ \left(1 + \sqrt{(\rho^{k,\lambda})^2 - \frac{b}{\lambda(a+b)}}\right)^2 \\ &+ \frac{b}{\lambda(a+b)},\end{aligned}$$

so that

$$P^{k,\lambda}(\theta) \leq \frac{a-b}{\lambda(a+b)} - \left(1 + \sqrt{(\rho^{k,\lambda})^2 - \frac{b}{\lambda(a+b)}}\right)^2. \tag{68}$$

Case $k \leq -1$. We assume now that $k \leq -1$. In this case, $\rho^{k,\lambda} < -\sqrt{\frac{b}{\lambda(a+b)}}$ and $\frac{du^{k,\lambda}}{d\theta}$ remains negative so that, using (62), we have:

$$\frac{du^{k,\lambda}}{d\theta} = -\sqrt{(\rho^{k,\lambda})^2 - \frac{b}{\lambda(a+b)}\sin^2 u^{k,\lambda}}.$$

So, for all $\theta \in \mathbb{R}$,

$$\rho^{k,\lambda} \leq \frac{du^{k,\lambda}}{d\theta}(\theta) \leq -\sqrt{(\rho^{k,\lambda})^2 - \frac{b}{\lambda(a+b)}} \tag{69}$$

and by integrating for $\theta \in [0, 2\pi]$, we obtain:

$$\rho^{k,\lambda} \leq k \leq -\sqrt{(\rho^{k,\lambda})^2 - \frac{b}{\lambda(a+b)}}, \tag{70}$$

so that

$$k^2 \leq (\rho^{k,\lambda})^2 \leq k^2 + \frac{b}{\lambda(a+b)}. \quad (71)$$

On the one hand, using (71) and that u' is negative,

$$(\rho^{k,\lambda})^2 + 2\frac{du^{k,\lambda}}{d\theta}(\theta) + 1 \leq k^2 + \frac{b}{\lambda(a+b)} + 1,$$

so:

$$P^{k,\lambda} \geq \frac{a-b}{\lambda(a+b)} - (k^2+1). \quad (72)$$

On the other hand, using (69), we obtain that:

$$(\rho^{k,\lambda})^2 + 2\frac{du^{k,\lambda}}{d\theta}(\theta) + 1 \geq (1+\rho^{k,\lambda})^2.$$

Now, from (70),

$$\rho^{k,\lambda} + 1 \leq k+1 \leq 0,$$

so

$$(\rho^{k,\lambda}+1)^2 \geq (k+1)^2.$$

Thus:

$$(\rho^{k,\lambda})^2 + 2\frac{du^{k,\lambda}}{d\theta}(\theta) + 1 \geq (k+1)^2.$$

Therefore, for all θ,

$$P^{k,\lambda}(\theta) \leq \frac{a}{\lambda(a+b)} - (k+1)^2. \quad (73)$$

B. Proof of Theorem 7

We fix $k \geq 1$. We remark that $\frac{a}{\lambda(a+b)} - \left(1+\sqrt{k^2+\frac{b}{\lambda(a+b)}}\right)^2$ is equivalent to $\frac{a-b}{\lambda(a+b)} > 0$ when $\lambda \longrightarrow 0$. So, using (67), for $\lambda > 0$ small enough, $P^{k,\lambda}$ is positive, thus, by Corollary 1, $M^0_{k,\lambda}$ is asymptotically stable modulo rotation-translation for (6).

When λ tends to $+\infty$, by (66), $\rho^{k,\lambda}$ tends to k, so

$$\frac{a-b}{\lambda(a+b)} - \left(1+\sqrt{(\rho^{k,\lambda})^2 - \frac{b}{\lambda(a+b)}}\right)^2 \longrightarrow -(1+k)^2 < 0.$$

Therefore, using (68), for λ large enough, $P^{k,\lambda}$ is bounded by a negative constant, so by Corollary 2, $M^{k,\lambda}$ is linearly unstable.(20).

Now we fix $k \leq -1$. Since $a > b$, when λ tends to zero,

$$\frac{a-b}{\lambda(a+b)} - (k+1)^2 \longrightarrow +\infty,$$

so, by Estimate 72, for λ small enough, we have:

$$\forall \theta \in \mathbb{R}, \; P^{k,\lambda}(\theta) > 0,$$

and using Corollary 1, we obtain that for $\lambda > 0$ small enough, $M^0_{k,\lambda}$ is asymptotically stable modulo rotation-translation.

When λ tends to $+\infty$,

$$\frac{a}{\lambda(a+b)} - (k+1)^2 \text{ tends to } -(k+1)^2.$$

Therefore, if $k \leq -2$, for λ large enough, using (73), $P^{k,\lambda}$ is negative, thus $M_0^{k,\lambda}$ is linearly unstable by Corollary 1.

This concludes the proof of Theorem 7.

C. Proof of Theorem 8

We fix $\lambda > 0$. From (65), when k tends to $+\infty$, $\rho^{k,\lambda}$ tends to $+\infty$. Therefore, with Estimate (68), for k large enough, for all θ, $P^{k,\lambda}(\theta)$ is negative, which implies that $M^{k,\lambda}$ is linearly unstable.

Using (73),for k in a neighbourhood of $-\infty$, for all θ, $P^{k,\lambda}(\theta)$ is negative, which implies that $M^{k,\lambda}$ is linearly unstable.

This concludes the proof of Theorem 8.

VIII. CONCLUSION

We described all the in-plane static configurations for the one-dimensional model of ferromagnetic ring (6), and we gave criteria to determine de stability of these solutions.

It could be interesting to describe the non-planar solutions. Some of them are evident: the magnetization constant equal to e_3 for example. A precise description of all these non-planar solutions could help us to prove that the solutions $\mathscr{M}_0^l$ and $M_0^{k,\lambda}$ are isolated. This could be of great help in demonstrating true instability results instead of linear instability theorems (see Theorems 6, 7, and 8).

Concerning the solutions $M_0^{k,\lambda}$, as it can be observed in numerical simulations, we conjecture that for all $\lambda > 0$, $M_0^{-1,\lambda}$ is asymptotically stable modulo rotation-translation.

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DATA AVAILABILITY

Data sharing not applicable – no new data generated

AUTHOR DECLARATIONS

CONFLICT OF INTEREST

The authors have no conflicts to disclose.

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