



# Sequential Equilibrium without Rational Expectations of Prices: A Theorem of Full Existence

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**SEQUENTIAL EQUILIBRIUM  
WITHOUT RATIONAL  
EXPECTATIONS OF PRICES:  
A THEOREM  
OF FULL EXISTENCE**

Lionel de BOISDEFFRE

SEQUENTIAL EQUILIBRIUM WITHOUT RATIONAL EXPECTATIONS OF PRICES:

A THEOREM OF FULL EXISTENCE

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(June 2017)

***Abstract***

*We consider a pure exchange economy, where agents, typically asymmetrically informed, exchange commodities, on spot markets, and securities of all kinds, on incomplete financial markets, with no model of how future prices are determined. They have private characteristics, anticipations and beliefs. We show they face an incompressible uncertainty, represented by a so-called "minimum uncertainty set", typically adding to the 'exogenous uncertainty', on tomorrow's state of nature, an 'endogenous uncertainty' on future spot prices, which may depend on every agent's private anticipations today. At equilibrium, all agents expect the 'true' price, in each realizable state, as a possible outcome, and elect optimal strategies, ex ante, which clear on all markets, ex post. Our main Theorem states that equilibrium exists as long as agents' prior anticipations, which may be refined from observing markets, embed that minimum uncertainty set. This result is stronger than the classical ones of generic existence, along Radner (1979), or Hart (1975), based on the rational expectation of prices.*

**Key words:** sequential equilibrium, temporary equilibrium, perfect foresight, existence, rational expectations, financial markets, asymmetric information, arbitrage.

**JEL Classification:** D52

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# 1 Introduction

When agents' information is incomplete or asymmetric, the issue of how markets may reveal information is essential, and yet debated. Quoting Ross Starr (1989), "*the theory with asymmetric information is not well understood at all. In short, the exact mechanism by which prices incorporate information is still a mystery and an attendant theory of volume is simply missing.*" A traditional response is given by the REE (rational expectations equilibrium) model by assuming, quoting Radner (1979), that "*agents have a 'model' or 'expectations' of how equilibrium prices are determined*". Under this assumption, agents know the relationship between private information signals and equilibrium prices, along a so-called "*forecast function*". Then, generically, prices reveal all information at a fully revealing REE equilibrium.

Cornet-De Boisdeffre (2002) suggests an alternative approach, where agents have no price model and asymmetric information is represented by private information signals, which correctly inform each agent that tomorrow's random state of nature will be in a subset of the state space. The model extends to this setting of asymmetric information the classical definitions of equilibrium, prices and arbitrage.

Generalizing Cass (1984) to asymmetric information, De Boisdeffre (2007) shows the existence of equilibrium on purely financial markets is, then, characterized by the no-arbitrage condition. This existence result is stronger than the REE's generic one, along Radner (1979). Moreover, Cornet-De Boisdeffre (2009) shows the no-arbitrage condition may be reached by agents with no price model, from simply observing exchange opportunities on financial markets.

Our above papers may picture the information transmission of actual markets and restore the full existence property of equilibrium. But they still retain Radner's

(1972) assumption that agents have a perfect foresight of future prices in each realizable state, which, quoting Radner (1982) himself "*seems to require of the traders a capacity for imagination and computation far beyond what is realistic*". Perfect foresight would be justified if agents knew all the primitives of the economy and their relationships with equilibrium prices, and, furthermore, elected one common anticipation (amongst typically many possibilities and opposite interests), with the common knowledge of game theory. The latter requirements are also referred to as rational expectations. Though standard, rational expectations rely on highly demanding assumptions and only yield the generic existence of sequential equilibrium, as shown, among others, by Radner (1979), for asymmetric information, or Hart (1975) and Duffie-Shaffer (1985), for symmetric information economies.

We propose to show that dropping rational expectations is, not only possible, but permits to reconcile into one unique concept the notions of sequential and temporary equilibria, and to insure the full existence of this so-called "*correct foresight equilibrium*", under milder conditions, and for any assets and information signals.

In the current paper, agents have no forecast function a la Radner and may keep their own characteristics private, which results in an incompressible uncertainty over future prices, represented by a so-called "*minimum uncertainty set*". We argue this set might be inferred from observing past prices and events. The model's sequential equilibrium, or "*correct foresight equilibrium*", is defined as Cornet-De Boisdeffre's (2002), but for price anticipations, which are now elements of private and typically uncountable sets. We assume, non restrictively along De Boisdeffre (2016), that agents' private anticipation sets preclude arbitrage, and show that equilibrium exists, if they include the minimum uncertainty set.

In our view, this approach to information transmission and equilibrium pictures

agents' actual behaviors on markets. Endowed with no price model and unaware of the primitives of the economy, they infer an arbitrage-free refinement of their information from observing trade, first. Whence reached, agents have no means of going beyond that refinement, and market forces, driven by prices, lead to equilibrium.

The paper is organized as follows: Section 2 presents the model. Section 3 states the existence Theorem and discusses its main Assumption. Section 4 proves the Theorem. An Appendix proves technical Lemmas.

## 2 The basic model

We consider, throughout, a two-period economy, with private information signals, a consumption market and a financial market. The sets,  $I$ ,  $S$ ,  $H$  and  $J$ , respectively, of consumers, states of nature, goods and assets are all finite. The first period will also be referred to as  $t = 0$  and the second, as  $t = 1$ . At  $t = 0$ , there is an uncertainty on which state of nature,  $s \in S$ , will prevail tomorrow. The non random state at  $t = 0$  is denoted by  $s = 0$  and, whenever  $\Sigma \subset S$ , we will denote  $\Sigma' := \{0\} \cup \Sigma$ .

### 2.1 Markets, information and beliefs

Agents consume and may exchange the same consumption goods,  $h \in H$ , on the spot markets of each period. The generic  $i^{th}$  agent's welfare is measured, ex post, by a utility index,  $u_i : \mathbb{R}_+^{2H} \rightarrow \mathbb{R}_+$ , over her consumptions at both dates.

At the first period, each agent,  $i \in I$ , receives some private information signal,  $S_i \subset S$ , about which states of the world may occur at  $t = 1$ . That is, she knows that no state,  $s \in S \setminus S_i$ , will prevail tomorrow. Each set  $S_i$  is assumed to contain the true state. Hence, the pooled information set, denoted by  $\underline{S} := \cap_{i \in I} S_i$ , is non-empty and

we let, w.l.o.g.,  $S = \cup_{i \in I} S_i$ . Such a collection of  $\#I$  finite sets, whose intersection is non-empty, is called an information structure.

Agents are unaware of the primitives of the economy (e.g., private characteristics or beliefs), hence, of how prices are determined. Therefore, they typically face uncertainty over future spot prices and, at  $t = 0$ , in each  $s \in S_i$ , the generic  $i^{th}$  agent has a private set of anticipations,  $P_s^i$ , of possible spot prices, a subset of  $P := \{p \in \mathbb{R}_{++}^H : \|p\| = 1\}$ <sup>2</sup>. Agents are, thus, concerned about relative prices only.

Throughout,  $\Omega_i := \cup_{s \in S_i} \{s\} \times P_s^i$  is given, for each agent  $i \in I$ , summing up her final uncertainty at  $t = 0$ , unless otherwise stated. The collection  $(\Omega_i)$  is an anticipation structure, along the following Definition, and we let  $\underline{\Omega} := \cap_{i \in I} \Omega_i$ . We refer to  $\Omega := S \times P$  as the forecast set and denote by  $\omega$  its generic element.

**Definition 1** *An anticipation set is a closed subset,  $Q$ , of  $\Omega := S \times P$ . An anticipation structure is a collection of anticipation sets,  $(Q_i)$ , such that:*

$$\forall s \in \underline{S}, (\{s\} \times P) \cap (\cap_{i \in I} Q_i) \neq \emptyset$$

*The set of anticipations structures is denoted by  $\mathcal{AS}$ .*

*Let  $(Q_i) \in \mathcal{AS}$  be given. An anticipation structure,  $(\underline{Q}_i) \in \mathcal{AS}$ , which is smaller (for the inclusion relation) than  $(Q_i)$ , is called a refinement of  $(Q_i)$ , and denoted by  $(\underline{Q}_i) \leq (Q_i)$ . It is said to be self-attainable if  $\cap_{i \in I} \underline{Q}_i = \cap_{i \in I} Q_i$ .*

*A belief is probability distribution over  $(\Omega, \mathcal{B}(\Omega))$ , whose support is an anticipation set.*

*A collection of beliefs,  $(\pi_i)$ , whose supports define the anticipation structure  $(Q_i) \in \mathcal{AS}$  is called a structure of beliefs, said to support  $(Q_i)$  and denoted by  $(\pi_i) \in \Pi_{(Q_i)}$ .*

Agents may operate financial transfers across states in  $S'$  by exchanging, at  $t = 0$ , finitely many assets  $j \in J$ , which pay off, at  $t = 1$ , conditionally on the realization of

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<sup>2</sup> As is standard,  $\mathbb{R}_+$  denotes the set of non-negative real numbers and  $\mathbb{R}_{++}$  denotes that of strictly positive.

forecasts. These payoffs, which may be nominal or real, define a continuous mapping,  $V : \Omega \rightarrow \mathbb{R}^J$ , which relates every forecast,  $\omega := (s, p) \in \Omega$  to the row vector,  $V(\omega) \in \mathbb{R}^J$ , of the  $\#J$  asset payoffs in units of account, conditionally on the joint occurrence of state  $s \in S$  and price  $p \in P$  on the spot market tomorrow. We denote by  $\mathcal{V}$  the above set of maps ( $V \in \mathcal{V}$ ) and let  $\mathcal{V}_\lambda := \{V' \in \mathcal{V} : \sup_{\omega \in \Omega} \|V'(\omega) - V(\omega)\| \leq \lambda\}$ , for every  $\lambda > 0$ .

Given the asset price,  $q \in \mathbb{R}^J$ , a portfolio,  $z = (z_j) \in \mathbb{R}^J$ , is a contract, which an agent may buy or sell at the cost of  $q \cdot z$  units of account at  $t = 0$ , and which specifies the quantities,  $z_j$ , of each asset  $j \in J$  (bought or sold) and delivers a flow,  $V(\omega) \cdot z$ , of conditional payoffs across forecasts,  $\omega \in \Omega$ .

**Definition 2** Given  $q \in \mathbb{R}^J$ , an anticipation structure,  $(Q_i) \in \mathcal{AS}$ , is said to be  $q$ -arbitrage-free if following no-arbitrage Condition holds:

$\nexists (i, z) \in I \times \mathbb{R}^J : -q \cdot z \geq 0$  and  $V(\omega) \cdot z \geq 0, \forall \omega \in Q_i$ , with one strict inequality.

A structure,  $(Q_i) \in \mathcal{AS}$ , is arbitrage-free if it is  $q$ -arbitrage-free for some  $q \in \mathbb{R}^J$ .

## 2.2 The agent's behaviour and the concept of equilibrium

Each agent,  $i \in I$ , receives an endowment,  $e_i := (e_{is}) \in \mathbb{R}_+^{HS'_i}$ , granting the commodity bundles,  $e_{i0} \in \mathbb{R}_+^H$  at  $t = 0$ , and  $e_{is} \in \mathbb{R}_+^H$ , in each state  $s \in S_i$ , if it prevails. We recall that agents' private forecasts are represented by an anticipation structure,  $(Q_i) \in \mathcal{AS}$ , which they have reached when they elect their strategies (with  $(Q_i) = (\Omega_i)$ , set as given, unless stated otherwise). Given  $(Q_i) \in \mathcal{AS}$  and the observed prices,  $\omega_0 := (p_0, q) \in \mathbb{R}_+^H \times \mathbb{R}^J$ , at  $t = 0$ , the generic  $i^{th}$  agent's consumption set is that of continuous mappings,  $x : Q'_i \rightarrow \mathbb{R}_+^H$  (where  $Q'_i := \{0\} \cup Q_i$ ), namely:  $X_i := \underline{\mathcal{C}}(Q'_i, \mathbb{R}_+^H)$ .

Thus, her consumptions,  $x \in X_i$ , are mappings, relating  $s = 0$  to a consumption decision,  $x_{\omega_0} := x_0 := (x_0^h) \in \mathbb{R}_+^H$ , at  $t = 0$ , and, continuously on  $Q_i$ , every forecast,

$\omega := (s, p_s) \in Q_i$ , to a consumption decision,  $x_\omega := (x_\omega^h) \in \mathbb{R}_+^H$ , at  $t = 1$ , which is conditional on the joint observation of state  $s$ , and price  $p_s$ , on the spot market. The generic  $i^{th}$  agent elects a strategy,  $(x, z) \in X_i \times \mathbb{R}^J$ , in the following budget set:

$$B_i(\omega_0, Q_i, V) := \{(x, z) \in X_i \times \mathbb{R}^J : p_0 \cdot (x_0 - e_{i0}) \leq -q \cdot z \text{ and } p_s \cdot (x_\omega - e_{is}) \leq V(\omega) \cdot z, \forall \omega := (s, p_s) \in Q_i\}.$$

Given agents' beliefs at the time of trading,  $(\pi_i) \in \Pi_{(Q_i)}$ , each consumer,  $i \in I$ , has preferences represented by the V.N.M. utility function:  $x \in X_i \mapsto U_i^{\pi_i}(x) := \int_{\omega \in Q_i} u_i(x_0, x_\omega) d\pi_i(\omega)$ .

This economy, denoted by  $\mathcal{E} = \{(I, S, H, J), V, (e_i)_{i \in I}, (u_i)_{i \in I}\}$ , retains the small consumer price-taker hypothesis, along which no single agent may, alone, have a significant impact on prices. It is called standard under the following Conditions:

- **Assumption A1** (*strong survival*): for each  $i \in I$ ,  $e_i \in \mathbb{R}_{++}^{HS'_i}$ ;
- **Assumption A2**: for each  $i \in I$ ,  $u_i$  is continuous, strictly concave and increasing:  $[(x, y, x', y') \in (\mathbb{R}_+^H)^4, (x, y) \leq (x', y'), (x, y) \neq (x', y')] \Rightarrow [u_i(x', y') > u_i(x, y)]$ ;
- **Assumption A3**: for every  $(i, h) \in I \times H$ , the mapping  $(x, y) \mapsto \partial u_i(x, y)/\partial y^h$  is defined and continuous on  $\{(x, y) \in \mathbb{R}_+^H \times \mathbb{R}_+^H : y^h > 0\}$ , and  $(\inf_A \partial u_i(x, y)/\partial y^h) > 0$ , for every bounded subset  $A \subset \{(x, y) \in \mathbb{R}_+^H \times \mathbb{R}_+^H : y^h > 0\}$ .

*Remark 1* In Assumption A2, which could be weakened, strict concavity is retained to alleviate the proof of a selection amongst optimal strategies (see the proof of Lemma 4 in the Appendix). Moreover, we notice the technical Assumption A3 is consistent with the standard Inada Conditions, without requiring them.

Consumers' behavior is to elect an optimal strategy in the buget set. So the below concept of equilibrium, which is both sequential, since all agents have self-fulfilling forecasts (under condition (a)), and temporary, since anticipations are exogenous.

**Definition 3** A collection of prices and forecasts,  $\omega_0 := (p_0, q) \in \mathbb{R}_+^H \times \mathbb{R}^J$  and  $\{\omega_s = (s, p_s)\}_{s \in \underline{\mathbf{S}}} \subset \underline{\mathbf{S}} \times P$ , of an anticipation structure,  $(Q_i) \in \mathcal{AS}$ , and supporting beliefs,  $(\pi_i) \in \Pi_{(Q_i)}$ , and of strategies,  $[(x_i, z_i)] \in \times_{i \in I} B_i(\omega_0, Q_i, V)$ , is said to be a (sequential) equilibrium of the economy,  $\mathcal{E}$ , or a correct foresight equilibrium (C.F.E.), if the following conditions hold:

- (a)  $\{\omega_s\}_{s \in \underline{\mathbf{S}}} \subset \underline{\mathbf{Q}} := \cap_{i \in I} Q_i$ ;
- (b)  $\forall i \in I, (x_i, z_i) \in \arg \max_{(x, z) \in B_i(\omega_0, Q_i, V)} U_i^{\pi_i}(x)$ ;
- (c)  $\sum_{i \in I} (x_{i\omega_s} - e_{is}) = 0, \forall s \in \underline{\mathbf{S}}'$ ;
- (d)  $\sum_{i \in I} z_i = 0$ .

Under above conditions, each forecast,  $\omega_s$  (for  $s \in \underline{\mathbf{S}}$ ), is said to support equilibrium.

For all  $\lambda > 0$ , we let  $\tilde{\mathcal{E}}_\lambda$  be the set of economies,  $\mathcal{E}_\lambda = \{(I, S, H, J), V_\lambda, (e_i), (u_i)\}$ , defined as above, with the only difference of the payoff map,  $V_\lambda \in \mathcal{V}_\lambda$ , replacing  $V \in \mathcal{V}$ , and whose equilibria are defined accordingly. Given  $\lambda > 0$ , we say that the economy,  $\mathcal{E}$ , admits a  $\lambda$ -equilibrium, if some economy,  $\mathcal{E}_\lambda \in \tilde{\mathcal{E}}_\lambda$ , has an equilibrium.

### 3 The existence theorem

We now show that uncertainty and existence of equilibrium are closely related.

#### 3.1 The minimum uncertainty set

**Definition 4** The minimum uncertainty set is the set,  $\Delta \subset \underline{\mathbf{S}} \times P$ , of forecasts, which support an equilibrium of the economy,  $\mathcal{E}$ , for some structure of beliefs today.

**Lemma 1** In a standard economy,  $\mathcal{E}$ , there exists  $\varepsilon \in ]0, 1[$ , such that  $\Delta \subset \underline{\mathbf{S}} \times [\varepsilon, 1]^H$ .

**Proof** See the Appendix. □

As shown below, in a standard economy,  $\Delta$  is never empty. This is due to the fact that, for every  $\lambda > 0$ , the economy,  $\mathcal{E}$ , admits a  $\lambda$ -equilibrium, from Theorem 1 of

our companion manuscript (see De Boisdeffre, 2017). Thus, for every  $\lambda > 0$ , the set,  $\Delta_\lambda$ , of forecasts which support a  $\lambda$ -equilibrium, is non empty. From the compactness of  $\Omega$ , the limit set  $\Delta^* := \lim_{\lambda \rightarrow 0^+} \Delta_\lambda = \cap_{n \in \mathbb{N}} \overline{\Delta_{1/n}}$  is also non-empty. The relation,  $\Delta \subset \Delta^*$ , holds from the Definition. We let the reader check from the Appendix that  $\Delta^*$  also meets the condition of Lemma 1. Leaving to subsequent research the study of whether the inclusion is strict, we now state our main Theorem.

**Theorem 1** *In a standard economy,  $\mathcal{E}$ , such that  $\Delta \subset \underline{\Omega}$ , an equilibrium exists, for any supporting beliefs,  $(\pi_i) \in \Pi_{(\Omega_i)}$ .*

Under the Theorem's Condition,  $\Delta \subset \underline{\Omega}$ , a CFE exists, for any beliefs,  $(\pi_i) \in \Pi_{(\Omega_i)}$ . We explain why  $\Delta$  is a set of "*minimum uncertainty*" and how it could be inferred.

On the first issue, when today's beliefs are private, no equilibrium price should be ruled out *a priori*. Theoretically, the set,  $\Delta$ , of all possible equilibrium prices tomorrow (for some beliefs today), is one of incompressible uncertainty. Practically, it could become so in times of enhanced uncertainty, volatility or erratic beliefs, which would prevent any agreement or visibility on the individual agents' forecasts.

On the second issue, the model specifies *normalised* prices. It is often possible to observe past prices and reckon their *relative* values, in a wide array of situations, or states, which typically replicate over time. Relative prices vary between observable upper and lower bounds. Along a sensible assumption, markets are mostly at equilibrium and, whenever price series are long enough, all equilibrium forecasts should lie within the bounds of the price series' convex hull. This statistical method and its iterative verification across periods require no price model and need not be performed by individuals, but by financial institutions, which could, e.g., on stock

markets, assess and predict plausible prices and sensible beliefs, from it.<sup>3</sup>

## 4 The existence proof

Throughout, we set as given arbitrary beliefs,  $(\pi_i) \in \Pi_{(\Omega_i)}$  and assume the economy,  $\mathcal{E}$ , is standard. We construct a sequence of auxiliary finite economies, tending to the initial one. All finite economies admit equilibria, whose sequence yields a C.F.E. Hereafter, we provisionally assume that  $\Delta^* \subset \underline{\Omega}$  (instead of  $\Delta \subset \underline{\Omega}$ ).

### 4.1 Finite partitions of agents' anticipation sets

- Let  $(i, n) \in I \times \mathbb{N}$  be given. We define a partition,  $\mathcal{P}_i^n = \{\Omega_{(i,n)}^k\}_{1 \leq k \leq K_{(i,n)}}$ , of  $\Omega_i$ , such that  $\pi_i(\Omega_{(i,n)}^k) > 0$ , for each  $k \leq K_{(i,n)}$ .
- In each set  $\Omega_{(i,n)}^k$  (for  $k \leq K_{(i,n)}$ ), we select exactly one interior element,  $\omega_{(i,n)}^k$ , forming the set,  $\Omega_i^n := \{\omega_{(i,n)}^k\}_{1 \leq k \leq K_{(i,n)}}$ .
- We define mappings,  $\pi_i^n : \Omega_i^n \rightarrow \mathbb{R}_+$ , by  $\pi_i^n(\omega_{(i,n)}^k) = \pi_i(\Omega_{(i,n)}^k)$  and  $\Phi_i^n : \Omega_i \rightarrow \Omega_i^n$ , by its restrictions,  $\Phi_i^n|_{\Omega_{(i,n)}^k}(\omega) = \omega_{(i,n)}^k$ , for each  $k \leq K_{(i,n)}$ .

**Lemma 2** For each  $i \in I$ , we may choose the above defined sequences,  $\{\mathcal{P}_i^n\}_{n \in \mathbb{N}}$ ,  $\{\Omega_i^n\}_{n \in \mathbb{N}}$  and  $\{\Phi_i^n\}_{n \in \mathbb{N}}$ , such that:

- (i) for every  $n \in \mathbb{N}$ ,  $\Omega_i^n \subset \Omega_i^{n+1}$  and  $\mathcal{P}_i^{n+1}$  is finer than  $\mathcal{P}_i^n$ ;
- (ii)  $\Omega_i = \overline{\lim_{n \rightarrow \infty} \nearrow \Omega_i^n} = \overline{\cup_{n \in \mathbb{N}} \Omega_i^n}$ , that is,  $\cup_{n \in \mathbb{N}} \Omega_i^n$  is everywhere dense in  $\Omega_i$ ;
- (iii) for every  $\omega \in \Omega_i$ ,  $\omega = \lim_{n \rightarrow \infty} \Phi_i^n(\omega)$ , and  $\Phi_i^n(\omega)$  converges uniformly to  $\omega$ .

**Proof** See the Appendix, which provides one example of such sequences.  $\square$

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<sup>3</sup> E.g., if the future reflects the past, if  $\underline{\mathbf{S}}$  is also a set of past states and, for every  $s \in \underline{\mathbf{S}}$ , the past price serie,  $(p_s^t) \in (P)^{T_s}$  (where  $T_s \in \mathbb{N}$ ) is large, the set  $\{(s, y_s) \in \underline{\mathbf{S}} \times P : y_s = \sum_{t=1}^{T_s} \alpha_t p_s^t / \| \sum_{t=1}^{T_s} \alpha_t p_s^t \|, (\alpha_t) \in \mathbb{R}_+^{T_s}, \sum_{t=1}^{T_s} \alpha_t = 1\}$ , could easily be checked, iteratively, to contain self-fulfilling forecasts, hence, assessed to contain  $\Delta$ .

## 4.2 The auxiliary economies, $\mathcal{E}^n$

Given  $n \in \mathbb{N}$ , we define a two-period economy,  $\mathcal{E}^n = \{(I, S, H, J), (\Theta_i^n), (e_i), (u_i)\}$ , where agents,  $i \in I$ , receive endowments,  $e_i \in \mathbb{R}_+^{HS'_i}$ , trade goods,  $h \in H$ , and assets,  $j \in J$ , as follows, for each  $i \in I$ :

- we let  $\Omega_i^{*n} := \{i\} \times \Omega_i^n$  and  $\Theta_i^n := \underline{\mathbf{S}} \cup \Omega_i^{*n}$  define an information structure,  $(\Theta_i^n)$ , of a formal state space,  $\Theta^n := \cup_{i \in I} \Theta_i^n$ , whose pooled information set is  $\underline{\mathbf{S}}$ .
- In each (realizable) state  $s \in \underline{\mathbf{S}}$ , the  $i^{th}$  agent is assumed to anticipate with perfect foresight the spot price to prevail.
- In each (purely formal) state  $(i, s, p) \in \Omega_i^{*n}$ , the agent has an idiosyncratic certainty that price  $p \in P$  will prevail.

For any price system  $(\omega_0 := (p_0, q), p := (p_s)) \in \mathbb{R}_+^H \times \mathbb{R}^J \times \mathbb{R}_+^{H\underline{\mathbf{S}}}$ , and payoff map,  $V^n \in \mathcal{V}_{1/n}$  (as defined in sub-Section 2.1), the agent's consumption set, budget set, and utility function are, respectively:

$X_i^n := \mathbb{R}_+^{H\underline{\mathbf{S}'}} \times \mathbb{R}_+^{H\Omega_i^n}$ , whose generic element is  $x := ((x_s)_{s \in \underline{\mathbf{S}}'}, (x_\omega)_{\omega \in \Omega_i^n})$ ;

$$\begin{aligned} \underline{B_i^n(\omega_0, p, V^n)} := & \{ (x, z) \in X_i^n \times \mathbb{R}^J : p_0 \cdot (x_0 - e_{i0}) \leq -q \cdot z \text{ and } p_s \cdot (x_s - e_{is}) \leq V^n(s, p_s) \cdot z, \forall s \in \underline{\mathbf{S}} \\ & \text{and } p \cdot (x_\omega - e_{is}) \leq V^n(s, p) \cdot z, \forall \omega := (s, p) \in \Omega_i^n \}; \end{aligned}$$

$$x \in X_i^n \mapsto u_i^n(x) := \sum_{s \in \underline{\mathbf{S}}} \frac{u_i(x_0, x_s)}{\mathcal{D}^{n+1} \# \underline{\mathbf{S}}} + (1 - \frac{1}{\mathcal{D}^{n+1}}) \sum_{\omega \in \Omega_i^n} u_i(x_0, x_\omega) \pi_i^n(\omega).$$

**Definition 5** *The collection of a payoff map,  $V^n \in \mathcal{V}_{1/n}$ , a price system,  $(\omega_0^n, p^n) \in \mathbb{R}_+^H \times \mathbb{R}^J \times \mathbb{R}_+^{H\underline{\mathbf{S}}}$ , and strategies,  $[(x_i^n, z_i^n)] \in \times_{i \in I} B_i^n(\omega_0^n, p^n, V^n)$ , is an equilibrium of the economy  $\mathcal{E}^n$  (and a  $\frac{1}{n}$ -equilibrium of the economy  $\mathcal{E}$ ) if the following conditions hold:*

$$(a) \forall i \in I, x_i^n \in \arg \max_{(x, z) \in B_i^n(\omega_0^n, p^n, V^n)} u_i^n(x);$$

$$(b) \sum_{i \in I} (x_{is}^n - e_{is}) = 0, \forall s \in \underline{\mathbf{S}}';$$

$$(c) \sum_{i \in I} z_i^n = 0.$$

From our companion paper's (op.cit.) Theorem 1 and proof, for every  $n \in \mathbb{N}$ , there exists  $V^n \in \mathcal{V}_{1/n}$ , for which the economy,  $\mathcal{E}^n$ , has an equilibrium,  $\mathcal{C}^n := ((\omega_0^n, p^n), V^n, [(x_i^n, z_i^n)]) \in (\mathbb{R}_+^H \times \mathbb{R}^J \times \mathbb{R}_+^{H\underline{\mathbf{S}}}) \times \mathcal{V}_{1/n} \times (\times_{i \in I} B_i^n(\omega_0^n, p^n))$ , such that  $\|p_s^n\| = 1$ , for each  $s \in \underline{\mathbf{S}}$ , and  $\|p_0^n\| + \|q^n\| = 1$ . These equilibria, set as given, satisfy Lemmas 3 &4.

**Lemma 3** *Let the sequence  $\{\mathcal{C}^n\}_{n \in \mathbb{N}}$ , be given from above. The following holds:*

- (i) *the sequence,  $\{(\omega_0^n, p^n)\}_{n \in \mathbb{N}}$ , may be assumed to converge, say to  $(\omega_0^*, p^*)$ , such that  $\|\omega_0^*\| = 1$  and  $\|p_s^*\| = 1$ , for every  $s \in \underline{\mathbf{S}}$ , and  $\{(s, p_s^*)\}_{s \in \underline{\mathbf{S}}} \subset \Delta^*$ ;*
- (ii) *the sequences  $\{(x_{is}^n)\}_{s \in \underline{\mathbf{S}}'}$  and  $\{(z_i^n)\}$  may be assumed to converge, say to  $(x_{is}^*)_{s \in \underline{\mathbf{S}}'}$ , and  $(z_i^*)$ , such that  $\sum_{i \in I} (x_{is}^* - e_{is})_{s \in \underline{\mathbf{S}}'} = 0$ , and  $\sum_{i \in I} z_i^n = 0$ .*

**Proof** see the Appendix. □

**Lemma 4** *Let  $B_i(\omega, z) = \{x \in \mathbb{R}_+^H : p \cdot (x - e_{is}) \leq V(\omega) \cdot z\}$ , for every  $(i, z, \omega := (s, p)) \in I \times \mathbb{R}^J \times \Omega$ , be given sets. Denote by  $\omega_s^* := (s, p_s^*)$ , and  $x_{i\omega_s^*}^* := x_{is}^*$ , for each  $(i, s) \in I \times \underline{\mathbf{S}}$ , the limits of Lemma 3. The following Assertions hold, for all  $i \in I$ :*

- (i) *for every  $s \in \underline{\mathbf{S}}$ ,  $\{x_{i\omega_s^*}^*\} = \arg \max u_i(x_{i0}^*, x)$ , for  $x \in B_i(\omega_s^*, z_i^*)$ ;*
- (ii) *the correspondence  $\omega \in \Omega_i \mapsto \arg \max u_i(x_{i0}^*, x)$ , for  $x \in B_i(\omega, z_i^*)$ , is a continuous mapping, whose embedding,  $x_i^* : \omega \in \Omega'_i \mapsto x_{i\omega}^*$ , defines a consumption plan;*
- (iii)  $U_i^{\pi_i}(x_i^*) = \lim_{n \rightarrow \infty} u_i^n(x_i^n)$ .

**Proof** see the Appendix. □

### 4.3 An equilibrium of the initial economy

We now prove Theorem 1, via the following Claim.

**Claim 1** *The collection,  $\{(\omega_s^*), (\pi_i), (x_i^*), (z_i^*)\}$ , of prices, forecasts, beliefs, allocation and portfolios of Lemmas 3-4, defines (jointly with  $(\Omega_i)$ ) a CFE of the economy  $\mathcal{E}$ .*

**Proof** Let us define  $\mathcal{C}^* := ((\omega_s^*), (\pi_i), [(x_i^*, z_i^*)])$  as in Claim 1. From Lemmas 3 and 4,  $\mathcal{C}^*$  meets Conditions (a)-(c)-(d) of Definition 3 of equilibrium above, since we have provisionally assumed that  $\Delta^* \subset \underline{\Omega}$ , while  $\{(s, p_s^*)\}_{s \in \underline{\mathbb{S}}} \subset \Delta^*$ , from Lemma 3. Thus, it suffices to show the joint relations  $[(x_i^*, z_i^*)] \in \times_{i \in I} B_i(\omega_0^*, \Omega_i, V)$  and Definition 3-(b).

First, we set  $i \in I$  as given, and show:  $(x_i^*, z_i^*) \in B_i(\omega_0^*, \Omega_i)$ . From the definition of  $\mathcal{C}^n$ , the relations  $p_0^n \cdot (x_{i0}^n - e_{i0}) \leq -q^n \cdot z_i^n$  hold, for each  $n \in \mathbb{N}$ , and, yield  $p_0^* \cdot (x_{i0}^* - e_{i0}) \leq -q^* \cdot z_i^*$ , in the limit. From Lemma 4, the relations  $x_i^* \in X_i$  and  $p_s \cdot (x_{i\omega}^* - e_{is}) \leq V(\omega) \cdot z_i^*$  also hold, for every  $\omega = (s, p_s) \in \Omega_i$ . Hence,  $[(x_i^*, z_i^*)] \in \times_{i \in I} B_i(\omega_0^*, \Omega_i, V)$  holds.

Next, we assume, by contraposition, that  $\mathcal{C}^*$  fails to meet Condition (b) of Definition 3, that is, there exist  $i \in I$ ,  $(x, z) \in B_i(\omega_0^*, \Omega_i, V)$  and  $\varepsilon \in \mathbb{R}_{++}$ , such that:

$$(I) \quad \varepsilon + U_i^{\pi_i}(x_i^*) < U_i^{\pi_i}(x).$$

$$\text{We may assume: } (II) \quad \exists (\delta, M) \in \mathbb{R}_{++}^2: x_\omega \in [\delta, M]^H, \forall \omega \in \Omega_i.$$

The existence of an upper bound to consumptions  $x_\omega$  (for  $\omega \in \Omega_i$ ) results from the relation  $(x, z) \in B_i(\omega_0^*, \Omega_i, V)$ , which implies a bound to financial transfers, and from the fact that  $\Omega_i$  is closed in  $S \times P$ . Moreover, for  $\alpha \in ]0, 1]$  small enough, the strategy  $(x^\alpha, z^\alpha) := ((1 - \alpha)x + \alpha e_i, (1 - \alpha)z) \in B_i(\omega_0^*, \Omega_i, V)$  meets both relations (I) and (II), from Assumption A1 and from the uniform continuity (on a compact set) of the mapping  $(\alpha, \omega) \in [0, 1] \times \Omega_i \mapsto u_i(x_0^\alpha, x_\omega^\alpha)$ . So, relations (II) may be assumed.

Then, we let the reader check, as immediate from the relations (I)-(II) and  $(x, z) \in B_i(\omega_0^*, \Omega_i, V)$ , from Lemma 3, the definition of  $\Omega_i$ , Assumptions A1-A2 and continuity arguments, that we may also assume there exists  $\gamma \in \mathbb{R}_{++}$ , such that:

$$(III) \quad p_0^*(x_0 - e_{i0}) \leq -q^* \cdot z \text{ and } p_s \cdot (x_\omega - e_{is}) \leq -\gamma + V(\omega) \cdot z, \forall \omega := (s, p_s) \in \Omega_i.$$

From relations (I)-(II)-(III), we may also assume there exists  $\gamma' \in ]0, \gamma[$ , such that:

$$(IV) \quad p_0^*(x_0 - e_{i0}) \leq -\gamma' - q^* \cdot z \text{ and } p_s \cdot (x_\omega - e_{is}) \leq -\gamma' + V(\omega) \cdot z, \forall \omega := (s, p_s) \in \Omega_i.$$

Indeed, the above assertion is obvious, from relations (III), if  $p_0^*(x_0 - e_{i0}) < -q^* \cdot z$ .

Assume that  $p_0^*(x_0 - e_{i0}) = -q^* \cdot z$ . If  $p_0^* = 0$ , then,  $q^* \neq 0$ , from Lemma 2-(iii), and relations (IV) hold if we replace  $z$  by  $z - q^*/N$ , for  $N \in \mathbb{N}$  big enough. If  $p_0^* \neq 0$  and  $x_0 \neq 0$ , the desired assertion results from Assumption A1 and above. Otherwise,  $-q^* \cdot z = -p_0^* \cdot e_{i0} < 0$ , and a slight change in portfolio insures relations (IV). From relations (IV), the continuity of the scalar product and Lemma 3, there exists  $N_1 \in \mathbb{N}$ , such that, for every  $n \geq N_1$ :

$$(V) \quad \left[ \begin{array}{l} p_0^n \cdot (x_0 - e_{i0}) \leq -q^n \cdot z \\ p_s^n \cdot (x_{\omega_s^*} - e_{is}) \leq V^n(s, p_s^n) \cdot z, \forall s \in \underline{\mathbf{S}} \\ p_s \cdot (x_\omega - e_{is}) \leq V^n(\omega) \cdot z = V(\omega) \cdot z, \forall \omega := (s, p_s) \in \Omega_i^n \end{array} \right].$$

Along relations (V), for each  $n \geq N_1$ , we define, in  $\mathcal{E}^n$ , the strategy  $(x^n, z) \in B_i^n(\omega_0^n, p^n, V^n)$  by  $x_0^n := x_0$ ,  $x_s^n := x_{\omega_s^*}$ , and  $x_{(i,\omega)}^n := x_\omega$ , for  $(s, \omega) \in \underline{\mathbf{S}} \times \Omega_i^n$ , and recall:

- $U_i^{\pi_i}(x) := \int_{\omega \in \Omega_i} u_i(x_0, x_\omega) d\pi_i(\omega);$
- $u_i^n(x^n) := \sum_{s \in \underline{\mathbf{S}}} \frac{u_i(x_0, x_s^n)}{\mathcal{Z}^{n+1} \# \underline{\mathbf{S}}} + (1 - \frac{1}{\mathcal{Z}^{n+1}}) \sum_{\omega \in \Omega_i^n} u_i(x_0, x_\omega) \pi_i^n(\omega).$

Then, from above, relation (II), Lemma 2, and the uniform continuity of  $x \in X_i$  and  $u_i$  on compact sets, there exists  $N_2 \geq N_1$  such that:

$$(VI) \quad |U_i^{\pi_i}(x) - u_i^n(x^n)| < \int_{\omega \in \Omega_i} |u_i(x_0, x_\omega) - u_i(x_0, x_{\Phi_i^n(\omega)})| d\pi_i(\omega) + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}, \text{ for every } n \geq N_2.$$

From equilibrium conditions and Lemma 4, there exists  $N_3 \geq N_2$ , such that:

$$(VII) \quad u_i^n(x^n) \leq u_i^n(x_i^n) < \frac{\varepsilon}{2} + U_i^{\pi_i}(x_i^*), \text{ for every } n \geq N_3.$$

Let  $n \geq N_3$  be given. The above Conditions (I)-(VI)-(VII) yield, jointly:

$$U_i^{\pi_i}(x) < \frac{\varepsilon}{2} + u_i^n(x^n) \leq \frac{\varepsilon}{2} + u_i^n(x_i^n) < \varepsilon + U_i^{\pi_i}(x_i^*) < U_i^{\pi_i}(x).$$

This contradiction proves that  $\mathcal{C}^*$  is a C.F.E. Indeed, from Lemma 3, the relation  $\{(s, p_s^*)\}_{s \in \underline{\mathbf{S}}} \subset \Delta^*$  holds, while  $\Delta^* \subset \underline{\Omega}$  was provisionally assumed. Yet, from above,  $\{(s, p_s^*)\}_{s \in \underline{\mathbf{S}}} \subset \Delta \subset \Delta^*$ , as a set of equilibrium forecasts. Hence, Theorem 1 holds.  $\square$

## Appendix

**Lemma 1** *In a standard economy,  $\mathcal{E}$ , there exists  $\varepsilon \in ]0, 1[$ , such that  $\Delta \subset \underline{\mathbf{S}} \times [\varepsilon, 1]^H$ .*

**Proof** Let a standard economy,  $\mathcal{E}^*$ , and a forecast,  $\omega := (s, p) \in \Delta$ , be given, which supports a CFE,  $\{(\omega_s)_{s \in \underline{\mathbf{S}}'}, (Q_i), V, (\pi'_i), [(x_i, z_i)]\}$ . The relation  $p := (p^h)_{h \in H} \in \mathbb{R}_{++}^H$  is standard from Assumption A2 and Definition 3-(b).

Let  $m := (\min_{(i, s, h) \in I \times \underline{\mathbf{S}} \times H} e_{is}^h) \in \mathbb{R}_{++}$  and  $M := (\max_{(s, h) \in \underline{\mathbf{S}}' \times H} \sum_{i \in I} e_{is}^h) \in \mathbb{R}_{++}$  be given, along Assumption A1. Then, the relations  $(x_{i0}) \geq 0$ ,  $(x_{i\omega}) \geq 0$ ,  $\sum_{i \in I} (x_{i0} - e_{i0}) = 0$  and  $\sum_{i \in I} (x_{i\omega} - e_{is}) = 0$ , which hold from Definition 2-(c), yield  $x_{i0} \in [0, M]^H$  and  $x_{i\omega} \in [0, M]^H$ , for each  $i \in I$ .

Let  $\alpha := \inf \partial u_i(x, y)/\partial y^h$ , for  $(i, h, (x, y)) \in I \times H \times \{(x, y) \in [0, M]^{2H} : y^h > 0\}$ , and  $\beta := \max \partial u_i(x, y)/\partial y^h$ , for  $(i, h, (x, y)) \in I \times H \times \{(x, y) \in [0, M]^{2H} : y^h \geq m\}$ , and  $\gamma = \beta/\alpha$  be strictly positive numbers, along Assumption A3, above.

Let  $(h, h') \in H^2$  be given and assume, by contraposition, that  $p^h/p^{h'} > \gamma$ . From the above relations, there exists at least one agent, say  $i = 1$ , unwilling to sell good  $h$ , when forecasting  $\omega := (s, p) \in \Delta$ , that is,  $x_{1\omega}^h \in [m, M]$ . We let the reader check, as tedious and standard, that agent  $i = 1$ , starting from  $(x_1, z_1)$ , could find a utility increasing strategy,  $(x_1^*, z_1) \in B_1(\omega_0, Q_1, V)$ , modifying her consumptions in her

forecast  $\omega$  only, such that  $x_{1\omega}^{*h} < x_{1\omega}^h$  and  $x_{1\omega}^{*h'} > x_{1\omega}^{h'}$ . Indeed, with  $p_s^h/p_s^{h'} > \gamma$ , she has an incentive to sell a small amount of the expensive commodity,  $h$ , in exchange for commodity  $h'$ . Hence,  $(x_1, z_1)$  cannot be an equilibrium strategy. This contradiction proves the relation  $p^h/p^{h'} \leq \gamma$ . We let the reader check, from the above relations,  $p > 0$ ,  $\|p\| = 1$  and  $p^h/p^{h'} \leq \gamma$ , for  $(h, h') \in H^2$ , that  $p^h \geq \varepsilon := 1/\gamma \# H$ , for each  $h \in H$ .  $\square$

**Lemma 2** *For each  $i \in I$ , we may choose the above defined sequences,  $\{\mathcal{P}_i^n\}_{n \in \mathbb{N}}$ ,  $\{\Omega_i^n\}_{n \in \mathbb{N}}$  and  $\{\Phi_i^n\}_{n \in \mathbb{N}}$ , such that:*

- (i) *for every  $n \in \mathbb{N}$ ,  $\Omega_i^n \subset \Omega_i^{n+1}$  and  $\mathcal{P}_i^{n+1}$  is finer than  $\mathcal{P}_i^n$ ;*
- (ii)  *$\Omega_i = \overline{\lim_{n \rightarrow \infty} \nearrow \Omega_i^n} = \overline{\cup_{n \in \mathbb{N}} \Omega_i^n}$ , that is,  $\cup_{n \in \mathbb{N}} \Omega_i^n$  is everywhere dense in  $\Omega_i$ ;*
- (iii) *for every  $\omega \in \Omega_i$ ,  $\omega = \lim_{n \rightarrow \infty} \Phi_i^n(\omega)$ , and  $\Phi_i^n(\omega)$  converges uniformly to  $\omega$ .*

**Proof** We set as given  $(i, n) \in I \times \mathbb{N}$ , recall, from Definition 1, that  $\Omega_i := \cup_{s \in S_i} \{s\} \times P_s^i$ , and let  $K^n := (\mathbb{N} \cap [1, 2^n])^H$ . For each  $s \in S_i$  and each  $k_n := (k_n^h) \in K^n$ , we define the set  $\Omega_i^{(s, k_n)} := \{s\} \times (P_s^i \cap \times_{h \in H} [\frac{k_n^h - 1}{2^n}, \frac{k_n^h}{2^n}])$ , and let  $K_s^n := \{k_n \in K^n : \pi_i(\Omega_i^{(s, k_n)}) > 0\}$ . The above sets yield ever finer partitions,  $\mathcal{P}_i^n := \{\Omega_i^{(s, k_n)}\}_{(s, k_n) \in S_i \times K_s^n}$ , of  $\Omega_i$ , for  $n \in \mathbb{N}$ .

For every triple  $(n, s, k_n) \in \mathbb{N} \times S_i \times K_s^n$ , we set as given one element,  $\omega_i^{(s, k_n)} \in \Omega_i^{(s, k_n)}$ , and construct sets  $\Omega_i^n := \{\omega_i^{(s, k_n)}\}_{(s, k_n) \in S_i \times K_s^n}$  such that  $\Omega_i^n \subset \Omega_i^{n+1}$ , for each  $n \in \mathbb{N}$ .

We define mappings,  $\Phi_i^n : \Omega_i \rightarrow \Omega_i^n$ , by  $\Phi_i^n(\omega) := \omega_i^{(s, k_n)}$ , for every tuple  $(n, s, k_n, \omega) \in \mathbb{N} \times S_i \times K_s^n \times \Omega_i^{(s, k_n)}$ , and finite probabilities,  $\pi_i^n$ , on  $\Omega_i^n$ , by the relations  $\pi_i^n(\omega_i^{(s, k_n)}) := \pi_i(\Omega_i^{(s, k_n)}) > 0$ , for every triple  $(n, s, k_n) \in \mathbb{N} \times S_i \times K_s^n$ . The above sequences,  $\{\mathcal{P}_i^n\}_{n \in \mathbb{N}}$ ,  $\{\Omega_i^n\}_{n \in \mathbb{N}}$ ,  $\{\Phi_i^n\}_{n \in \mathbb{N}}$  and  $\{\pi_i^n\}_{n \in \mathbb{N}}$  (for  $i \in I$ ) satisfy the properties of Lemma 2.  $\square$

**Lemma 3** *Let the sequence  $\{\mathcal{C}^n\}_{n \in \mathbb{N}}$ , be given from above. The following holds:*

- (i) *the sequence,  $\{(\omega_0^n, p^n)\}_{n \in \mathbb{N}}$  may be assumed to converge, say to  $(\omega_0^*, p^*)$ , such that  $\|\omega_0^*\| = 1$  and  $\|p^*\| = 1$ , for every  $s \in \underline{\mathbf{S}}$ , and  $\{(s, p_s^*)\}_{s \in \underline{\mathbf{S}}} \subset \Delta^*$ ;*

(ii) the sequences  $\{(x_{is}^n)_{s \in \underline{\mathbf{S}}'}\}$  and  $\{(z_i^n)\}$  may be assumed to converge, say to  $(x_i^*)_{s \in \underline{\mathbf{S}}'}$ , and  $(z_i^*)$ , such that  $\sum_{i \in I} (x_{is}^* - e_{is})_{s \in \underline{\mathbf{S}}'} = 0$ , and  $\sum_{i \in I} z_i^n = 0$ .

**Proof** Assertion (i) is obvious from continuity and compactness arguments.  $\square$

Assertion (ii) The non-negativity and market clearance conditions over consumptions imply that consumptions are bounded in each state  $s \in \underline{\mathbf{S}}'$ , hence, may be assumed to converge. To prove that attainable strategies are also bounded, we first let  $n \in \mathbb{N}$  be given and show attainable strategies of the economy  $\mathcal{E}^n$  are bounded. From the definition of anticipations, it suffices to show that portfolios are bounded.

- Let  $\delta = \|(e_i)\|$ . Assume, by contraposition, that, for every  $k \in \mathbb{N}$ , there exist attainable strategies,  $[(x_i^k, z_i^k)]$ , in the economy  $\mathcal{E}^n$ , and prices,  $(p_s^k) \in P^{\underline{\mathbf{S}}}$ , such that  $\|z^k\| := \|(z_i^k)\| > k$ . For each  $k \in \mathbb{N}$ , market clearance and budget conditions yield:

$$\sum_{i \in I} z_i^k = 0, \text{ and } V^n(s, p_s^k) \cdot z_i^k \geq -\delta, \quad \forall (i, s, k) \in I \times \underline{\mathbf{S}} \times \mathbb{N}.$$

- For every  $(i, k) \in I \times \mathbb{N}$ , we let  $x_i'^k := \frac{x_i^k}{\|z^k\|} + (1 - \frac{1}{\|z^k\|})e_i$  and  $z_i'^k := \frac{z_i^k}{\|z^k\|}$ . Then,  $\|(z_i'^k)\| = 1$  and the sequences,  $\{(z_i'^k)\}_{k \in \mathbb{N}}$  and  $\{(p_s^k)\}_{k \in \mathbb{N}}$ , have cluster points,  $(z_i)$ , and  $(p_s)$ , such that  $\|(z_i)\| = 1$ , and:

$$\begin{aligned} \sum_{i \in I} z_i'^k &= 0, \quad V^n(s, p_s^k) \cdot z_i'^k \geq -\delta/k, \quad \forall (i, s, k) \in I \times \underline{\mathbf{S}} \times \mathbb{N}, \text{ and, passing to the limit,} \\ \sum_{i \in I} z_i &= 0, \quad V^n(s, p_s) \cdot z_i \geq 0, \quad \forall (i, s) \in I \times \underline{\mathbf{S}}, \end{aligned}$$

From our companion paper (op.cit.), by construction, the truncation to  $\underline{\mathbf{S}}$  of  $V^n$  has full column rank and the latter relations imply  $(z_i) = 0$  and contradict the fact that  $\|(z_i)\| = 1$ . This contradiction proves that attainable strategies are bounded in every finite number of economies. We now prove that they are bounded across any number of economies.

- As above, we need only show portfolios are bounded. We let the reader check that all contraposition arguments above translate, mutatis mutandis, and en-

large to the following ones, on double indexed sequences of prices,  $(p_s^{(n,k)})$ , and portfolios  $(z_i^{(n,k)})$ , where  $(n, k) \in \mathbb{N}^2$  ( $n$  standing for the economy). The final contraposition arguments are (with same notations):

$$\sum_{i \in I} z_i'^{(n,k)} = 0, V^n(s, p_s^{(n,k)}) \cdot z_i'^{(n,k)} \geq -\delta/k,$$

$$V^n(\omega_i) \cdot z_i'^{(n,k)} \geq -\delta/k, \forall (i, s, \omega_i, n, k) \in I \times \underline{\mathbf{S}} \times \Omega_i^n \times \mathbb{N}^2.$$

- Given  $n \in \mathbb{N}$ , by construction of  $V^n$  along our companion paper (op.cit.), the payoffs of  $V$  and  $V^n$  only differ in states  $s \in \underline{\mathbf{S}}$ . So, we may write the above inequalities in fictitious states only as follows:

$$\sum_{i \in I} z_i'^{(n,k)} = 0, V(\omega_i) \cdot z_i'^{(n,k)} \geq -\delta/k, \forall (i, \omega_i, n, k) \in I \times \Omega_i^n \times \mathbb{N}^2$$

- For every  $(i, n) \in I \times \mathbb{N}$ , we let  $Z_i := \{z \in \mathbb{R}^j : V(\omega)z = 0, \forall \omega \in \Omega_i\}$ , and  $Z_i^\perp$  be its orthogonal,  $Z_i^n := \{z \in \mathbb{R}^j : V(\omega)z = 0, \forall \omega \in \Omega_i^n\}$  and  $Z_o := \sum_{i \in I} Z_i$ . We let the reader check that  $Z_i^n := Z_i$ , for  $n \in \mathbb{N}$  large enough, so we will assume w.l.o.g. that  $Z_i^n := Z_i$  for every  $(i, n) \in I \times \mathbb{N}$ . Taking the projections on the orthogonals, the above relations may be written:

$$\sum_{i \in I} z_i'^{(n,k)} \in Z_o, V(\omega_i) \cdot z_i'^{(n,k)} \geq -\delta/k, \forall (i, \omega_i) \in I \times \Omega_i^n \text{ with } z_i'^{(n,k)} \in Z_i^\perp,$$

hence, in the limit,  $\sum_{i \in I} z_i \in Z_o$ ,  $V(\omega_i) \cdot z_i \geq 0$ ,  $\forall (i, \omega_i) \in I \times \Omega_i$ , with  $\|(z_i)\| = 1$ .

- The latter relations imply (because  $(\Omega_i)$  is arbitrage-free)  $(z_i) \in \times_{i \in I} Z_i$ , hence,  $(z_i) \in \times_{i \in I} Z_i \cap Z_i^\perp = \{0\}$ , which contradicts the relation  $\|(z_i)\| = 1$ . This contradiction completes the proof of Assertion (ii). The rest of the Assertion is immediate from market clearance conditions on equilibria, passing to the limit.  $\square$

**Lemma 4** Let  $B_i(\omega, z) = \{x \in \mathbb{R}_+^H : p \cdot (x - e_{is}) \leq V(\omega) \cdot z\}$ , for every  $(i, z, \omega := (s, p)) \in I \times \mathbb{R}^J \times \Omega$ , be given sets. Denote by  $\omega_s^* := (s, p_s^*)$ , and  $x_{i\omega_s^*}^* := x_{is}^*$ , for each  $(i, s) \in I \times \underline{\mathbf{S}}$ , the limits of Lemma 3. The following Assertions hold, for all  $i \in I$ :

- (i) for every  $s \in \underline{\mathbf{S}}$ ,  $\{x_{i\omega_s^*}^*\} = \arg \max u_i(x_{i0}^*, x)$ , for  $x \in B_i(\omega_s^*, z_i^*)$ ;
- (ii) the correspondence  $\omega \in \Omega_i \mapsto \arg \max u_i(x_{i0}^*, x)$ , for  $x \in B_i(\omega, z_i^*)$ , is a continuous mapping, whose embedding,  $x_i^* : \omega \in \Omega_i \mapsto x_{i\omega}^*$ , defines a consumption plan;
- (iii)  $U_i^{\pi_i}(x_i^*) = \lim_{n \rightarrow \infty} u_i^n(x_i^n)$ .

### Proof

Assertion (i) Let  $(i, s) \in I \times \underline{\mathbf{S}}$  be given. For each  $n \in \mathbb{N}$ , the fact that  $\mathcal{C}^n$  is an equilibrium of  $\mathcal{E}^n$  implies:  $x_{is}^n \in \arg \max_{y \in B_i^n(\omega_s^n, z_i^n)} u_i(x_{i0}^n, y)$ , where  $B_i^n(\omega, z) = \{x \in \mathbb{R}_+^H : p \cdot (x - e_{is}) \leq V^n(\omega) \cdot z\}$ .

As standard from Berge's Theorem (see, e.g., Debreu, 1959, p. 19), with slight abuse the correspondence (mapping from Assumption A2),  $(x, \omega, z, V^n) \in \mathbb{R}_+^H \times \Omega \times \mathbb{R}^J \times \mathcal{V} \mapsto \arg \max_{y \in B_i^n(\omega, z)} u_i(x, y)$ , is continuous at  $(x_{i0}^*, \omega_s^*, z_i^*, V)$ , since  $u_i$  and  $B_i^n$  are. Moreover, from above,  $(x_{i0}^*, x_{is}^*, \omega_s^*, z_i^*, V) = \lim_{n \rightarrow \infty} (x_{i0}^n, x_{is}^n, \omega_s^n, z_i^n, V^n)$ . Hence, the latter relations (for  $n \in \mathbb{N}$ ) pass to limit and yield:  $\{x_{i\omega_s^*}^*\} := \{x_{is}^*\} = \arg \max_{y \in B_i(\omega_s^*, z_i^*)} u_i(x_{i0}^*, y)$ .  $\square$

Assertion (ii) Let  $i \in I$  and  $B_i^n(\omega, z) = \{x \in \mathbb{R}_+^H : p \cdot (x - e_{is}) \leq V^n(\omega) \cdot z\}$ , for each  $n \in \mathbb{N}$ , be given. For every  $(\omega, n) \in \Omega_i \times \mathbb{N}$ , the fact that  $\mathcal{C}^n$  is an equilibrium of  $\mathcal{E}^n$  and Assumption A2 imply:  $\{x_{i\Phi_i^n(\omega)}^n\} = \arg \max_{y \in B_i^n(\Phi_i^n(\omega), z_i^n)} u_i(x_{i0}^n, y)$ .

By the same token, the mapping,  $(x, \omega, z, V^n) \in \mathbb{R}_+^H \times \Omega \times \mathbb{R}^J \times \mathcal{V} \mapsto \arg \max_{y \in B_i^n(\omega, z)} u_i(x, y)$ , is continuous. Moreover, from above, the relation  $(x_{i0}^*, \omega, z_i^*, V) = \lim_{n \rightarrow \infty} (x_{i0}^n, \Phi_i^n(\omega), z_i^n, V^n)$  holds. Hence, the above relations (for  $n \in \mathbb{N}$ ) pass to the limit and yield a continuous mapping,  $\omega \in \Omega_i \mapsto x_{i\omega}^* := \arg \max_{y \in B_i(\omega, z_i^*)} u_i(x_{i0}^*, y)$ , whose embedding,  $x_i^* : \omega \in \{0\} \cup \Omega_i \mapsto x_{i\omega}^*$ , is a consumption plan,  $x_i^* \in X_i$ , from the definition.  $\square$

Assertion (iii) Let  $i \in I$  be given and  $x_i^* \in X_i$  be defined from above. With slight abuse in notations, let  $\varphi_i : (x, \omega, z, V^n) \in \mathbb{R}_+^H \times \Omega_i \times \mathbb{R}^J \times \mathcal{V} \mapsto \arg \max_{y \in B_i^n(\omega, z)} u_i(x, y)$  be defined on its domain. By the same token,  $\varphi_i$  and  $U_i : (x, \omega, z, V^n) \mapsto u_i(x, \varphi_i(x, \omega, z, V^n))$

are continuous and, moreover, the relations  $u_i(x_{i0}^*, x_{i\omega}^*) = U_i(x_{i0}^*, \omega, z_i^*, V)$  and  $u_i(x_{i0}^n, x_{i\Phi_i^n(\omega)}^n) = U_i(x_{i0}^n, \Phi_i^n(\omega), z_i^n, V^n)$  hold, for every  $(\omega, n) \in \Omega_i \times \mathbb{N}$ . Then, the uniform continuity of  $u_i$  and  $U_i$  on compact sets, yield, from above:

$$(I) \quad \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} : \forall n > N_\varepsilon, \forall \omega \in \Omega_i, |u_i(x_{i0}^*, x_{i\omega}^*) - u_i(x_{i0}^n, x_{i\Phi_i^n(\omega)}^n)| < \varepsilon.$$

Moreover, we recall the following definitions, for every  $n > N$ :

$$(II) \quad U_i^{\pi_i}(x_i^*) := \int_{\omega \in \Omega_i} u_i(x_{i0}^*, x_{i\omega}^*) d\pi_i(\omega);$$

$$(III) \quad u_i^n(x_i^n) := \sum_{s \in \underline{\mathbf{S}}} \frac{u_i(x_{is}^n, x_{is}^n)}{2^{n+1} \# \underline{\mathbf{S}}} + (1 - \frac{1}{2^{n+1}}) \sum_{\omega \in \Omega_i^n} u_i(x_{i0}^n, x_{i\omega}^n) \pi_i^n(\omega).$$

Then, Assertion (iii) results immediately from relations (I)-(II)-(III) above.  $\square$

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