Equilibrium in Incomplete Markets with Differential Information: A Basic Model of Generic Existence

Lionel Boisdeffre

W o r k i n g   P a p e r s   S e r i e s

Centre d'Analyse Théorique et de Traitement des données économiques

Center for the Analysis of Trade and economic Transitions

CATT WP No. 12
August 2018

EQUILIBRIUM IN INCOMPLETE MARKETS WITH DIFFERENTIAL INFORMATION: A BASIC MODEL OF GENERIC EXISTENCE

Lionel de BOISDEFFRE

CATT-UPPA
Collège Sciences Sociales et Humanités (SSH)
Droit, Economie, Gestion
Avenue du Doyen Poplawski - BP 1633
64016 PAU Cedex - FRANCE
Tél. (33) 5 59 40 80 61/62
Internet : **http://catt.univ-pau.fr/live/**

UNIVERSITÉ DE PAU ET DES PAYS DE L'ADOUR

EQUILIBRIUM IN INCOMPLETE MARKETS WITH DIFFERENTIAL INFORMATION:

A BASIC MODEL OF GENERIC EXISTENCE

Lionel de Boisdeffre,[1]

(August 2018)

Abstract

The paper demonstrates the generic existence of general equilibria in incomplete markets with asymmetric information. The economy has two periods and an ex ante uncertainty over the state of nature to be revealed at the second period. Securities pay off in cash or commodities at the second period, conditionally on the state of nature to be revealed. They permit financial transfers across periods and states, which are insufficient to span all state contingent claims to value, whatever the spot price to prevail. Under smooth preference and the standard Radner (1972) perfect foresight assumptions, we show that equilibria exist, except for a closed set of measure zero of endowments and securities. This result extends Duffie-Shafer's (1985) in three ways. First, it allows for asymmetric information amongst agents. Second, it holds whenever the equilibrium price is given a fixed norm on each spot market. Third, assets need no longer pay off in commodities, but also in any mix of cash and goods.

Key words: sequential equilibrium, temporary equilibrium, perfect foresight, existence, rational expectations, financial markets, asymmetric information, arbitrage.

JEL Classification: D52

[1] University of Paris 1-Panthéon-Sorbonne, 106-112 Bd. de l'Hôpital, 75013 Paris. Email address: lionel.de.boisdeffre@wanadoo.fr

1 Introduction

This paper demonstrates, with standard arguments, the generic existence of equilibrium in incomplete financial markets with differential information. It presents a two-period pure exchange economy, with an ex ante uncertainty over the state of nature to be revealed at the second period. Asymmetric information is represented by private finite subsets of states, which each agent is correctly informed to contain the realizable states. Consumers exchange consumption goods on spot markets, and, unrestrictively, assets of any kind on financial markets. They are endowed with a bundle of goods in every state, with ordered smooth preferences over their consumptions, and with a perfect foresight of future prices, along Radner (1972).

A companion paper, dropping perfect foresight, provides conditions which insure the full existence of equilibrium when agents exchange real assets. The current paper, however, studies existence on arbitrary financial markets under the perfect foresight assumption. Its generic existence result is classical and weaker than De Boisdeffre's (2007), where financial markets are purely nominal. The latter paper, generalizing Cass (1984) to asymmetric information, shows that the full existence of equilibrium with nominal assets is characterized, in the current model, by the absence of unlimited arbitrage opportunity on financial markets. Along De Boisdeffre (2016), that no-arbitrage condition can always be achieved, with agents having no price model, from their observing available transfers on financial markets.

When assets pay off in goods, equilibrium needs not exist, as shown by Hart (1975) in the symmetric information case. His example is based on the collapse of the span of assets' payoffs, that occurs at clearing prices. In our model, an additional problem arises from differential information. Financial markets may be arbitragefree for some commodity prices, and not for others, in which case equilibrium cannot exist. We show this problem vanishes owing to a good property of payoff matrices.

Attempts to resurrect the existence of equilibrium with real assets noticed that the above "*bad*" prices could only occur exceptionally, as a consequence of Sard's theorem. These attempts include Mc Manus (1984), Repullo (1984), Magill & Shafer (1984, 1985), for potentially complete markets (i.e., complete for at least one price), and Duffie-Shafer (1985, 1986), for incomplete markets. These papers apply to symmetric information, build on differential topology arguments, and demonstrate the generic existence of equilibrium, namely, existence except for a closed set of measure zero of economies, parametrized by the assets' payoffs and agents' endowments.

The current paper is highly indebted to Duffie-Shafer (1985), to which several claims on Grassmanians and differential topology borrow. It extends the latter paper in three ways. First, it allows for asymmetric information amongst consumers. Second, financial structures cover any mix of nominal and real assets, whereas Duffie-Shafer (1985) deals with real assets and symmetric information only. Third, it normalizes (to arbitrary values) the equilibrium price on every spot market. In Duffie-Shafer (1985), only the value of one particular consumer's endowment is normalized to one across all states of nature. Duffie-Shafer's purpose is to prove the existence of equilibrium under the perfect foresight assumption. So, the relevance and the means of inferring equilibrium prices are no issues.

In the current paper, however, normalizing price anticipations in every state of nature to relevant values is an important issue, because it is a step towards dropping the perfect foresight assumption, also called the rational expectation assumption. This standard assumption, made by Radner (1972), which states that agents know the map between the state of nature and the spot price to obtain, is seen as unrealistic by most theorists, including Radner (1982) himself. But no definition of a sequential equilibrium was given so far, which dropped the assumption. In a companion paper, we show that dropping rational expectations is not only possible, but also a means of restoring the full existence property of sequential equilibrium, in all financial and information structures. Then, with no model to forecast future prices with certainty, agents need restrict their expectations to relevant normalized prices in every state, which the current paper permits.

We make use of the standard differential topology arguments, introduced by Debreu (1970, 1972) for the study of general equilibrium. Following Duffie-Shafer (1985), we define a so-called "*pseudo-equilibrium*" with asymmetric information and a related concept of equilibrium. We derive the full existence of pseudo-equilibria from modulo 2 degree theory and manifolds' properties. Then, Sard's theorem serves to prove that pseudo-equilibria generically coincide with equilibria.

The paper is organized as follows: Section 2 presents the model, defines the concepts of equilibrium and pseudo-equilibrium and provides the main properties of Grassmanians. Section 3 presents the pseudo-equilibrium manifold and its properties. Section 4 states and proves the existence theorems.

2 The model

Throughout the paper, we consider a pure-exchange economy with two periods, $t \in \{0, 1\}$, and an uncertainty, at $t = 0$, upon which state of nature will randomly prevail, at $t = 1$. Consumers exchange goods, on spots markets, and assets of all kinds, on typically incomplete financial markets, independently of the spot price to prevail. The sets, I, S, L and J, respectively, of consumers, states of nature, consumption goods and assets are all finite. The state of the first period ($t = 0$) is denoted by $s = 0$ and we let $\Sigma' := \{0\} \cup \Sigma$, for every subset, Σ, of S. Similarly, $l = 0$ denotes the unit of account and we let $L' := \{0\} \cup L$.

We present information signals and markets, in sub-Section 2.1, consumer's behaviour and the concept of equilibrium, in sub-Section 2.2, a related concept of pseudo-equilibrium in sub-Section 2.3. For expositional purposes, we resume and summarize matrices' properties and the model's notations in the last sub-Sections.

2.1 Markets and information

Agents consume or exchange the consumption goods, $l \in L$, on both periods' spot markets. At $t = 0$, each agent, $i \in I$, receives privately some correct information signal, $S_i \subset S$ (henceforth given), that the true state will be in S_i. We assume costlessly that $S = \cup_{i \in I} S_i$. Thus, the pooled information set, $\underline{\mathbf{S}} := \cap_{i \in I} S_i$, containing the true state, is non-empty, and the relation $\underline{\mathbf{S}} = S$ characterizes symmetric information.

Commodity prices, $p \in \mathbb{R}^L$, on any future spot market, will be restricted to the unit hemisphere, $\Delta := \{p \in \mathbb{R}^L_{++} : \|p\| = 1\}$. Normalization to one is assumed for convenience but non restrictive. In any state, that bound could be replaced by any positive value. Since no state from the set $S \backslash \underline{\mathbf{S}}$ may prevail, we assume that each agent, $i \in I$, forms an idiosyncratic anticipation, $p_i := (p_{is}) \in \Delta^{S_i \backslash \underline{\mathbf{S}}}$ of spot prices in such states, if $S_i \neq \underline{\mathbf{S}}$. To alleviate subsequent definitions and notations, we assume w.l.o.g. that $p_{is} = p_{js} := \overline{p}_s$ holds, for any pair of agents, $(i, j) \in I^2$, anticipating state $s \in S_i \cap S_j \backslash \underline{\mathbf{S}}$. We refer to $P := \{\ p := (p_s) \in \Delta^S :\ p_s = \overline{p}_s,\ \forall s \in S \backslash \underline{\mathbf{S}}\ \}$, and $\Omega := S \times \Delta$ as, respectively, the price anticipation set and the forecast set.

Agents may operate transfers across states in S' by exchanging, at $t = 0$, finitely many assets, $j \in J$, which pay off, at $t = 1$, conditionally on the realization of forecasts. We will assume that $\#J \leqslant \#\underline{\mathbf{S}}$, so as to cover incomplete markets. These conditional payoffs may be nominal or real or a mix of both. The generic payoffs of an asset, $j \in J$, in a state, $s \in S$, are a bundle, $v_j(s) := (v_j^l(s)) \in \mathbb{R}^{L'}$, of the quantities, $v_j^0(s)$, of cash, and $v_j^l(s)$, of each good $l \in L$, delivered if state s prevails. All payoffs define a $(S \times L') \times J$ payoff matrix, V, which is identified (with same notation) to a continuous map, $V : \Omega \to \mathbb{R}^J$, relating the forecasts, $\omega := (s, p) \in \Omega$, to the rows, $V(\omega) \in \mathbb{R}^J$, of all assets' payoffs in cash, delivered if state s and price p obtain. At asset price, $q \in \mathbb{R}^J$, agents may buy or sell unrestrictively portfolios, $z = (z_j) \in \mathbb{R}^J$, for $q \cdot z$ units of account at $t = 0$, against the promise of delivery of a flow, $V(\omega) \cdot z$, of conditional payoffs across forecasts, $\omega \in \Omega$.

For notational purposes, we let $\mathcal{V}$ be the set of $(\underline{\mathbf{S}} \times L') \times J$ payoff matrices, defined, mutatis mutandis, as the matrix V above, equiped with the same notations and with the Euclidean norm and related topology. For every $p := (p_s) \in P$, and every $V' \in \mathcal{V}$, we let $V'(p)$ be the $\underline{\mathbf{S}} \times J$ matrix, whose generic row is $V'(s, p_s)$ (for $s \in \underline{\mathbf{S}}$). We now state a Claim, which will serve later.

Claim 1 *Let $p := (p_s) \in P$ and $V' \in \mathcal{V}$ be such that $\text{rank}\ V'(p) = \#J$. The following Assertions hold:*

(i) $\nexists (z_i) \in \mathbb{R}^{J \times I} \backslash \{0\} : \sum_{i \in I} z_i = 0$ *and* $V'(s, p_s) \cdot z_i \geqslant 0,\ \forall (i, s) \in I \times \underline{\mathbf{S}}$*;*

(ii) $\exists q \in \mathbb{R}^J,\ \forall i \in I,\ \exists \lambda_i := (\lambda_{is}) \in \mathbb{R}^{S_i}_{++} : q = \sum_{s \in \underline{\mathbf{S}}} \lambda_{is} V'(s, p_s) + \sum_{s \in S_i \backslash \underline{\mathbf{S}}} \lambda_{is} V(s, p_s)$.

Proof Assertion (i): Let $(z_i) \in \mathbb{R}^{J \times I}$ be such that $\sum_{i \in I} z_i = 0$ and $V'(s_i, p_{s_i}) \cdot z_i \geqslant 0$ for each pair $(i, s_i) \in I \times \underline{\mathbf{S}}$. These relations imply that $V'(s, p_s) \cdot z_i = 0$, for each $i \in I$, and each $s \in \underline{\mathbf{S}}$. Since $V'(p)$ has full rank, the latter relations imply $(z_i) = 0$. □

Assertion (ii) is a direct consequence of Assertion (i), above, from Cornet-De

Boisdeffre's (2002) Lemma 1, p. 398, and Proposition 3.1, p. 401. □

2.2 The consumer's behaviour and concept of equilibrium

Each agent, $i \in I$, receives an endowment, $e_i := (e_{is})$, granting the commodity bundles, $e_{i0} \in \mathbb{R}^L_{++}$ at $t = 0$, and $e_{is} \in \mathbb{R}^L_{++}$, in each expected state, $s \in S_i$, if it prevails. Given prices, $p := (p_s) \in \mathbb{R}^L_{++} \times P$, for commodities, and $q \in \mathbb{R}^J$, for securities, the generic i^{th} agent's consumption set is $X_i := \mathbb{R}^{L \times S'_i}_{++}$ and her budget set is:

$B_i(p, q) := \{\ (x, z) \in X_i \times \mathbb{R}^J : p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot z \ \ and \ \ p_s \cdot (x_s - e_{is}) \leqslant V(s, p_s) \cdot z, \ \forall s \in S_i\ \};$[2]

Each consumer, $i \in I$, has preferences represented by a utility function, $u_i : X_i \to \mathbb{R}$ and optimises her consumption in the budget set. The above economy is denoted by $\mathcal{E} = \{(I, S, L, J), V, (S_i)_{i \in I}, (e_i)_{i \in I}, (u_i)_{i \in I}\}$ and yields the following equilibrium concept:

Definition 1 *A collection of prices,* $p \in \mathbb{R}^L_{++} \times P$ *and* $q \in \mathbb{R}^J$*, and strategies,* $(x_i, z_i) \in B_i(p, q)$*, defined for each* $i \in I$*, is an equilibrium of the economy,* $\mathcal{E}$*, if the following conditions holds:*

(a) $\forall i \in I, \ (x_i, z_i) \in \arg\max u_i(x)$, *for* $(x, z) \in B_i(p, q)$*;*

(b) $\sum_{i \in I} (x_{is} - e_{is}) = 0, \ \forall s \in \underline{\mathbf{S}}'$*;*

(c) $\sum_{i \in I} z_i = 0$.

The economy, $\mathcal{E}$, is called standard if it meets the following conditions:

Assumption A1 : $\forall i \in I, \ u_i \ \ is \ \ C^\infty$;

Assumption A2 : $\forall i \in I, \ u_i$ *satisfies the Inada Conditions*;

Assumption A3 : $\forall i \in I, \forall x \in X_i, \ Du_i(x) \in X_i$ *(strict monotonicity)*;

Assumption A4: $\forall i \in I, \ h^T D^2 u_i(x) h < 0, \ \forall h \neq 0, \ h \cdot Du_i(x) = 0$;

[2] As in Duffie-Shafer (1985), our existence proof could not avoid the artefactual interior consumptions at equilibrium. Dropping them may be a next step of research.

Assumption A5: *$\forall i \in I$, $\forall \overline{x} \in X_i$, $\{ x \in X_i : u_i(x) \geqslant u_i(\overline{x}) \}$ is closed in X_i;*

Assumption A6: *one agent, say $i = 1$, has full information, i.e., $S_1 = \underline{\mathbf{S}}$.*

2.3 A related concept of pseudo-equilibrium

We now define a related concept of pseudo-equilibrium, after the following sets:

- we let $\mathcal{G}$ be the set of $\underline{\mathbf{S}} \times J$ full column rank matrices;
- for every $\Sigma \subset S$ and every $\Sigma \times J$ matrix L, $< L >$ denotes the span of L in $\mathbb{R}^{\Sigma}$;
- for every triple $(p, i, x) \in \mathbb{R}^{L}_{++} \times P \times I \times X_i$, we let $p \,\Box_i\, x \in \mathbb{R}^{S_i}$ be the vector, whose generic component is the scalar product, $p_s \cdot x_s$, for $s \in S_i$;
- for every $i \in I$ and every $L := (L_s) \in \mathcal{G}$, we let $L^i := (L^i_s)$ be the $S_i \times J$ matrix, whose generic row is $L^i_s := L_s$, for $s \in \underline{\mathbf{S}}$, and $L^i_s := V(s, \overline{p}_s)$, for $s \in S_i \backslash \underline{\mathbf{S}}$.

We define the concept of pseudo-equilibrium of the economy, $\mathcal{E}$, as follows:

Definition 2 *The collection of a scalar, $y \in \mathbb{R}_{++}$, prices, $p := (p_s) \in \mathbb{R}^{L}_{++} \times P$, payoff matrices, $V' \in \mathcal{V}$, and $L \in \mathcal{G}$, consumptions, $x_i := (x_{is}) \in X_i$, endowments, $e'_i := (e'_{is}) \in X_i$, for each $i \in I$, define a pseudo-equilibrium of the economy $\mathcal{E}$ if the following conditions hold:*

(a) $x_1 \in \arg\max\ u_i(x)$, *for* $x \in \{ x \in X_1 \ : \sum_{s \in S'_1} p_s \cdot (x_s - e'_{1s}) = 0 \}$;

(b) *for every* $i \in I \backslash \{1\}$, $x_i \in \arg\max\ u_i(x)$,

for $x \in \{ x \in X_i \ : \ \sum_{s \in S'_i} p_s \cdot (x_s - e'_{is}) = 0$ *and* $p \,\Box_i\, (x - e'_i) \in < L^i > \}$;

(c) $< V'(p) > \subset < L >$;

(d) $\sum_{i \in I} (x_{is} - e_{is}) = 0, \ \forall s \in \underline{\mathbf{S}}'$;

(e) $\sum_{s \in S'_1} p_s \cdot e'_{1s} = y$.

Given $(e', V') \in (\times_{i \in I} X_i) \times \mathcal{V}$, we say that $(p, L) \in \mathbb{R}^{L}_{++} \times P \times \mathcal{G}$ is a pseudo-equilibrium, if there exists $(x, y) \in (\times_{i \in I} X_i) \times \mathbb{R}_{++}$, such that $(x, y, p, < L >, e', V')$[3] is a pseudo-

[3] *With slight abuse we will also denote (x, y, p, L, e', V') a pseudo-equilibrium.*

equilibrium along Conditions (a) to (e), above. We let $\mathcal{E}^*$ be the pseudo-equilibrium manifold, or the set of collections, (p, L, e', V'), such that (p, L) is a pseudo-equilibrium, given (e', V'). We define the projection, $\pi : (p, L, e', V') \in \mathcal{E}^* \mapsto (e', V') \in (\times_{i \in I} X_i) \times \mathcal{V}$.

Remark 1: We chose to define pseudo-equilibria and equilibria with reference to financial structures mixing both nominal and real assets. This is no restriction. All arguments and results of the current paper hold if assets pay off in goods (or in cash) only. For nominal assets, the full existence of equilibrium in this model is demonstrated in De Boisdeffre (2007), extending Cass (1984).

Claim 2 *Let (x, y, p, L, e', V') be a pseudo-equilibrium. Then, there exists $(z_i) \in \mathbb{R}^{J \times I}$, such that:*

(i) $\sum_{i \in I} z_i = 0$;

(ii) $\forall (i, s) \in I \times S_i, \ p_s \cdot (x_{is} - e'_{is}) = L^i_s \cdot z_i$

Proof Condition (b) of Definition 2, yields:

$$\forall i \in I \backslash \{1\}, \ \exists z_i \in \mathbb{R}^J : \forall s \in S_i, \ p_s \cdot (x_{is} - e'_{is}) = L^i_s \cdot z_i.$$

Let $z_1 := -\sum_{i \in I \backslash \{1\}} z_i$. Then, $(z_i) \in \mathbb{R}^{J \times I}$ meets Assertion (i) of Claim 2. From Condition (d) of Definition 2, for every $s \in \underline{\mathbf{S}}$, $p_s \cdot (x_{1s} - e'_{1s}) = -\sum_{i \in I \backslash \{1\}} p_s \cdot (x_{is} - e'_{is}) = -\sum_{i \in I \backslash \{1\}} L_s \cdot z_i = L_s \cdot z_1$. Hence, from Assumption *A6*, Assertion (ii) holds. □

2.4 The Grassmanian's main properties

The notion and properties of the pseudo-equilibrium rely heavily on those of the set $\mathcal{G}$, henceforth called, with slight abuse, Grassmanian.[4] We therefore recall this set's main properties, in particular, the following Claim 3.

[4] In the standard definition, the Grassmanian is the set of $\#J$-dimensional subspaces of $\mathbb{R}^{\underline{\mathbf{S}}}$, an equivalent definition through the span.

Claim 3 *The Grassmanian, $\mathcal{G}$, is a C^∞ compact manifold without boundary of dimension $v^{**} := \#J(\#\underline{\mathbf{S}} - \#J)$.*

Proof The proof is given in Duffie-Shafer (1985, Fact 3, p. 292). □

To be more specific about Claim 3, we now summarize some standard results on Grassmanians, referring to Duffie-Shafer (1985) for details. Those who are unfamiliar with differential topology and manifolds may refer to Milnor (1997).

Let $\mathcal{W}$ be the open set of $(\underline{\mathbf{S}} - J) \times \underline{\mathbf{S}}$ matrices of rank $(\#\underline{\mathbf{S}} - \#J)$ and $L \in \mathcal{G}$ be given. We say that $W \in \mathcal{W}$ induces L, and we write it $W \in L$, if the product of the two matrices satisfies $WL = 0 \in \mathbb{R}^{\underline{\mathbf{S}} \times \underline{\mathbf{S}}}$. We notice that, if $W \in L$, then, $W' \in L$, if and only if there exists a non-singular $(\underline{\mathbf{S}} - J) \times (\underline{\mathbf{S}} - J)$ matrix A, such that $W' = AW$. This condition is written $W \sim W'$ and defines equivalence classes on $\mathcal{W}$. By the relation $WL = 0$, we may identify the Grasmanian, $\mathcal{G}$, to the set of equivalence classes, $\mathcal{W}/\sim$, endowed with the quotient topology. That is, U is open in $\mathcal{W}/\sim$ if and only if $p^{-1}(U)$ is open in $\mathcal{W}$, where $p : \mathcal{W} \to \mathcal{W}/\sim$ is the identification map.

On $\mathcal{W}$ we define the set, Σ, of permutations of the lines of matrices (e.g., inverting or unchanging the ranks of those lines, are permutations). For every $\sigma \in \Sigma$, we denote by P_σ the $\underline{\mathbf{S}} \times \underline{\mathbf{S}}$ permutation matrix corresponding to σ. Then, it is shown that every element of $\mathcal{W}/\sim$ is represented by a $(\underline{\mathbf{S}} - J) \times \underline{\mathbf{S}}$ matrix of the form $[I|E]P_\sigma$, for some permutation, $\sigma \in \Sigma$, where I is the $(\underline{\mathbf{S}} - J) \times (\underline{\mathbf{S}} - J)$ identity matrix and E is a unique $(\underline{\mathbf{S}} - J) \times J$ matrix. Notations are obvious: $[I|E]$ is a $(\underline{\mathbf{S}} - J) \times \underline{\mathbf{S}}$ matrix, whose first columns are those of I, followed by those of E. Sated differently:

the relation $L \in \mathcal{G}$ (identified to $\mathcal{W}/\sim$) holds if and only if there exists $\sigma \in \Sigma$ and one $(\underline{\mathbf{S}} - J) \times J$ matrix, E, such that $[I|E]P_\sigma \in L$

For every permutation, $\sigma \in \Sigma$, this yields the following sets and mappings:

$W_\sigma := \{L \in \mathcal{G} : \exists E \in \mathbb{R}^{(\underline{\mathbf{S}}-J)\times J},\ [I|E]P_\sigma \in L\}$

and $\varphi_\sigma : W_\sigma \to \mathbb{R}^{(\underline{\mathbf{S}}-J)\times J}$ defined by $[I|\varphi_\sigma(L)]P_\sigma \in L$.

Then, Claim 3 results from Duffie-Shafer's more specific results as follows:

Claim 4 *The following Assertions hold:*

(i) $\{W_\sigma\}_{\sigma\in\Sigma}$ *is an open cover of* $\mathcal{G}$*;*

(ii) φ_σ *is a homeomorphism;*

(iii) $\varphi_\sigma \circ \varphi_{\sigma'}^{-1} : \varphi_{\sigma'}(W_\sigma \cap W_{\sigma'}) \to \varphi_\sigma(W_\sigma \cap W_{\sigma'})$ *is smooth for all* σ, σ'*;*

(iv) $\mathcal{G}$ *is compact.*

Proof See Duffie-Shafer (1985, Fact 3, p. 292). □

For every $\sigma \in \Sigma$, we define the map $K_\sigma : \mathbb{R}^L_{++} \times P \times W_\sigma \times \mathbb{R}^{v^*} \to \mathbb{R}^{(\underline{\mathbf{S}}-J)\times\underline{\mathbf{S}}}$ by $K_\sigma(p, L, V') = [I|\varphi_\sigma(L)]P_\sigma V'(p)$ and we recall the following properties:

Claim 5 *The following Assertions hold:*

(i) K_σ *is* C^∞*;*

(ii) the partial derivative $D_{V'}(p, L, V')$ *has rank* $v^{**} := (\#\underline{\mathbf{S}} - \#J)\#J$.

Proof See Duffie-Shafer (1985, Fact 7, p. 294). □

2.5 The model's notations

For convenience, we gather on a single page all model's notations:

- $\mathcal{E} = \{(I, S, L, J), V, (S_i)_{i\in I}, (e_i)_{i\in I}, (u_i)_{i\in I}\}$ summarizes the economy's characteristics. There are two periods, $t \in \{0, 1\}$, finite sets, I, S, L, J, respectively, of consumers, states, goods and assets, a payoff matrix, V, information sets, $S_i \subset S$, and $\underline{\mathbf{S}} := \cap_{i\in I} S_i \neq \varnothing$, endowments, e_i, and utilities, u_i, defined for each $i \in I$.

- We let $s = 0$ be the state at $t = 0$, $l = 0$ be the account unit and denote $L' := \{0\} \cup L$, $\underline{\mathbf{S}}' := \{0\} \cup \underline{\mathbf{S}}$, $S_i' := \{0\} \cup S_i$, consumption sets, $X_i := \mathbb{R}_{++}^{L \times S_i'}$, for $i \in I$.

- $\Delta := \{p \in \mathbb{R}_{++}^L : \|p\| = 1\}$ and $\Omega := S \times \Delta$ are sets of anticipations and forecasts.

- $P := \{ \ p := (p_s) \in \Delta^S : \ p_s = \overline{p}_s, \ \forall s \in S \backslash \underline{\mathbf{S}} \ \}$ (with $(\overline{p}_s) \in \mathbb{R}_{++}^{L \times S \backslash \underline{\mathbf{S}}}$ exogenously given).

- $\mathcal{V}$ is the set of $(\underline{\mathbf{S}} \times L') \times J$ matrices and $\mathcal{G}$ is that of full rank $\underline{\mathbf{S}} \times J$ matrices.

- $V(s, p_s) \in \mathbb{R}^J$ denotes the row of payoffs in cash of V, if the forecast $(s, p_s) \in \Omega$ obtains. The notation extends to the elements of $\mathcal{V}$.

- $V'(p)$, for $(p, V') \in P \times \mathcal{V}$, is the $\underline{\mathbf{S}} \times J$ matrix of generic row $V'(s, p_s) \in \mathbb{R}^J$, for $s \in \underline{\mathbf{S}}$.

- For every $\Sigma \subset S$ and every $\Sigma \times J$ matrix V', $< V' >$ denotes the span of V' in $\mathbb{R}^\Sigma$.

- For every triple $(p, i, x) \in \mathbb{R}_{++}^L \times P \times I \times X_i$, we let $p \, \square_i \, x \in \mathbb{R}^{S_i}$ be the vector, whose generic component is the scalar product, $p_s \cdot x_s$, for $s \in S_i$.

- For every $i \in I$ and every $L := (L_s) \in \mathcal{G}$, we let $L^i := (L_s^i)$ be the $S_i \times J$ matrix, whose generic row is $L_s^i := L_s$, for $s \in \underline{\mathbf{S}}$, and $L_s^i := V(s, \overline{p}_s)$, for $s \in S_i \backslash \underline{\mathbf{S}}$.

- Futhemore, we let $l^* := (\#L + \#\underline{\mathbf{S}}(\#L - 1))$ be the dimension of the price set, $\mathbb{R}_{++}^L \times P$, we let $v^* := \#\underline{\mathbf{S}}\#L'\#J$, be that of the financial structure, $v^{**} := (\#\underline{\mathbf{S}} - \#J)\#J$ be that of $\mathcal{G}$, and $e^* := \sum_{i \in I} \#S_i'\#L$ be that of all agents' endowments.

3 The pseudo-equilibrium manifold

For the first agent, $i = 1$, we define the demand correspondence, $G_1 : \mathbb{R}_{++}^{l^*+1} \to X_1$, by $G_1(y, p) := \arg\max \ u_i(x)$, for $x \in \{ \ x \in X_1 \ : \sum_{s \in S_1'} \ p_s \cdot x_s = y \ \}$. In the latter problem, $y > 0$ is taken as given. As classical results, in a standard economy, G_1 is a C^∞ map, such that, given y, $\lim_{p \to \overline{p}} \|G_1(y, p)\| = +\infty$ whenever $\overline{p} \in \partial(\mathbb{R}_{++}^{l^*}) \backslash \{0\}$.

For all other agents, $i \in I\backslash\{1\}$, we define the demand correspondence, $D_i : \mathbb{R}^{l^*}_{++} \times \mathcal{G} \times X_i \ \to X_i$, by $D_i(p, L, e'_i) := \arg\max\ u_i(x)$, for $x \in \{\ x \in X_i\ :\ \sum_{s \in S'_i}\ p_s \cdot (x_s - e'_{is}) = 0\ \ and\ \ p_{\ \square_i}\ (x - e'_i)\ \in < L^i > \}$. In a standard economy, D_i is also a C^∞ map.

Then, we make use of Walras' law, which is possible from Claim 2, above. We pick up one good, say $l = 1$. For every $i \in I$, and every consumption $x_i \in X_i$, we denote by $x^*_i \in \mathbb{R}^{l^*}_{++}$ the truncation of x_i obtained by eliminating the good $l = 1$ from all spot markets at $t = 1$, and eliminating spot markets in all unrealizable states, $s \in S\backslash\underline{\mathbf{S}}$. We notice that $l^* := (\#L + \#\underline{\mathbf{S}}(\#L - 1))$ is the total number of spot markets left after truncation, as well as the dimension of the price manifold, $\mathbb{R}^L_{++} \times P$. We denote similarly (with stars) the truncations of the above demands. Given $(y, p, L, e' := (e'_i)) \in \mathbb{R}^{l^*+1}_{++} \times \mathcal{G} \times \mathbb{R}^{e^*}_{++}$, the excess demand:

$$Z(y, p, L, (e'_i)) := G^*_1(y, p) + \sum_{i \in I\backslash\{1\}}\ D^*_i(p, L, e'_i) - \sum_{i \in I}\ e'^*_i$$

defines a demand correspondence, $Z : \mathbb{R}^{l^*+1}_{++} \times \mathcal{G} \times \mathbb{R}^{e^*}_{++} \ \to \mathbb{R}^{l^*}$. It follows from above that Z is a C^∞ map, whose (partial) derivative satisfies $D_{e^*_1} Z(y, p, L, (e'_i)) = -I$, where I stands for the $l^* \times l^*$ identity matrix. We notice from the limit property of G_1 that $\lim_{(y,p,L,e') \to (\overline{y}, \overline{p}, \overline{L}, \overline{e}')} \|Z(y, p, L, e')\| = +\infty$ whenever $(\overline{y}, \overline{p}, \overline{L}, \overline{e}') \in \mathbb{R}_{++} \times \partial(\mathbb{R}^{l^*}_{++})\backslash\{0\} \times \mathcal{G} \times \mathbb{R}^{e^*}_{++}$.

Let $h : \mathbb{R}^{l^*+1}_{++} \times X_1 \to \mathbb{R}$ be the map defined by $h(y, p, e'_1) := p \cdot e'_1 - y$. We recall the definitions and properties of sub-Section 2.4 and define, for every $\sigma \in \Sigma$, the map $H_\sigma : \mathbb{R}^{l^*+1}_{++} \times W_\sigma \times \mathbb{R}^{e^*}_{++} \times \mathbb{R}^{v^{**}} \to \mathbb{R}^{l^*+1} \times \mathbb{R}^{v^{**}}$ by $H_\sigma(y, p, L, e', V') := (h(y, p, e'_1), Z(y, p, L, e'), K_\sigma(p, L, V'))$. Then, it follows from the definitions, and Claim 4, that the pseudo-equilibrium manifold is $\mathcal{E}^* = \cup_{\sigma \in \Sigma} H^{-1}_\sigma(0)$. We now show the following properties:

Claim 6 *For each $\sigma \in \Sigma$, 0 is a regular value of H_σ.*

Proof Let $\sigma \in \Sigma$ be given. By the same token as Duffie-Shafer's (1985, Fact 8, p. 294), we consider the derivative of H_σ with respect to y, e'^*_1 and V':

$$D_{(y,e_1'^*,V')}H_\sigma(y,p,L,e',V') = \begin{pmatrix} D_y h(y,p,e_1') = -1 & 0 & 0 \\ 0 & D_{e_1^*} Z(y,p,L,e') = -I & 0 \\ 0 & 0 & D_{V'}(p,L,V') \end{pmatrix}.$$

From Claim 5, this matrix has rank $1 + l^* + v^{**}$. Claim 6 follows. □

Claim 7 *$\mathcal{E}^*$ is a submanifold of $\mathbb{R}^{l^*+1}_{++} \times \mathcal{G} \times \mathbb{R}^{e^*}_{++} \times \mathbb{R}^{v^*}$ without boundary of dimension $e^* + v^*$.*

Proof The proof is due to Duffie-Shafer (1985, fact 9, p. 295). We just have to add one map, h, and one variable, y, and anticipate from Section 4 that $\mathcal{E}^*$ is non-empty. The argument is as follows: from Claim 6, its proof, and the pre-image theorem, for each $\sigma \in \Sigma$, the set $H_\sigma^{-1}(0)$ is a submanifold of $\mathbb{R}^{l^*+1}_{++} \times W_\sigma \times \mathbb{R}^{e^*}_{++} \times \mathbb{R}^{v^*}$ (hence, of $\mathbb{R}^{l^*+1}_{++} \times \mathcal{G} \times \mathbb{R}^{e^*}_{++} \times \mathbb{R}^{v^*}$) of dimension $(l^*+1+v^{**}+e^*+v^*) - (1+l^*+v^{**}) = e^* + v^*$. Then, Claim 7 results from the relation $\mathcal{E}^* = \cup_{\sigma\in\Sigma} H_\sigma^{-1}(0)$, which holds from above. □

Claim 8 *The following Assertions hold:*

(i) *the projection map, $\pi : \mathcal{E}^* \to \mathbb{R}^{e^*}_{++} \times \mathbb{R}^{v^*}$, is proper, that is, the inverse image by π of a compact set is compact;*

(ii) *there exists a regular value (e^*, V^*) of π, such that $\#\pi^{-1}(e^*, V^*) = 1$;*

Proof Assertion (i) The proof is the same (up to the addition of the variable $y > 0$) as Duffie-Shafer's (1985, Fact 10, p. 295), which the reader is invited to read. □

Assertion (ii) We set as given a price, $p^* := (p_s^*) \in P$, and a matrix, $V^* \in \mathcal{V}$, such that $V^*(p^*) \in \mathcal{G}$, and we let $L^* := V^*(p^*)$. We choose $V^*(p^*)$, such that the first $\#J^{th}$ rows are linearly independent. From Assumption *A2*, we choose endowments,

$e^* := (e_i^*) \in \mathbb{R}^{e^*}_{++}$, which make each agent's gradient, $\nabla u_i(e_i^*)$, for $i \in I$, colinear to prices, $(p_s^*)_{s \in S_i'}$. By construction, in a standard economy, the collection $(p^*, V^*, (e_i^*))$ defines a pure spot pseudo-equilibrium with no trade and it is Pareto optimal. Hence, there are no infinitesimal portfolios $(z_i) \in \mathbb{R}^{J \times I}$, along Claim 2, which permit to Pareto improve the allocation (e_i^*).

Assume, by contraposition, that there exists another pseudo-equilibrium in the set $\pi^{-1}(e^*, V^*)$. Assume, first, that it is not a pure spot one. Then, from Assumption *A4*, there exist mutually improving transfers, along Claim 2, relative to (e_i^*). From above, such improving transfers do not exist, so, the (other) pseudo-equilibrium is a pure spot market one. Since prices are fixed in all unrealizable states, $s \notin \underline{\mathbf{S}}$, the (optimal) pseudo-equilibrium allocations, (e_{is}^*), will, hence, not change in those states. Since (e_i^*) is Pareto optimal and affordable at any price, $\mathcal{C} := ((e_i^*), y^* = (p_s^*)_{\underline{\mathbf{S}}} \cdot e_1^*, p^*, < L^* >, (e_i^*), V^*)$ is the only pseudo-equilibrium in $\pi^{-1}(e^*, V^*)$. Thus, $\#\pi^{-1}(e^*, V^*) = 1$.

We now check that (e^*, V^*) is a regular value of π. Since the current model's prices and payoffs are all fixed in all unrealizable states ($s \notin \underline{\mathbf{S}}$), the proof is the same as Duffie-Shafer's (1985, p. 296), to which we refer the reader. □

4 The existence Theorems

We start with the full existence property of pseudo-equilibria.

Theorem 1 *For every payoff matrix, $V' \in \mathcal{V}$, and every collection of endowments, $e' := (e_i') \in \mathbb{R}^{e^*}_{++}$, the economy, $\mathcal{E}$, admits a pseudo-equilibrium, $(x, y, p, L, e', V') \in \mathbb{R}^{e^*}_{++} \times \mathbb{R}^{l^*+1}_{++} \times \mathcal{G} \times \mathbb{R}^{e^*}_{++} \times \mathbb{R}^{v^*}$, along Definition 2, above.*

Proof It is a standard application of mod 2 degree theory to the map π: if $f : X \to Y$ is a smooth proper map between two boundaryless manifolds of the same dimension, with Y connected, the number, $\#f^{-1}(y)$, of elements $x \in X$, such that $y = f(x)$, is the same, modulo 2, for every regular value $y \in Y$. In particular, if one regular value, y, of f, is such that $\#f^{-1}(y)$ is odd, then, $f^{-1}(y)$ is non-empty for every $y \in Y$. Indeed, y is by definition regular if $f^{-1}(y) = \varnothing$. From Claims 7 and 8, the map, π, meets all desired properties, for $X := \mathcal{E}^*$ and $Y := \mathbb{R}^{e^*}_{++} \times \mathbb{R}^{v^*}$, and yields Theorem 1.$\square$

Let $\mathcal{R}_\pi$ be the set of regular values of π and $\mathcal{R}^c_\pi$ be its complement. At any regular value, (e', V'), there exists $(x, y, p, L, e', V') \in \mathcal{E}^*$, such that $< V'(p) >=< L >$. As standard from Sard's theorem (see Milnor, p. 10), $\mathcal{R}^c_\pi$ is of zero Lebesgue measure. Since π is proper, $\mathcal{R}^c_\pi$ is also closed.

For every $V' \in \mathcal{V}$, we henceforth let $\widetilde{V}'$ be the $(S \times L') \times J$ matrix, which coincides with V' on $\underline{\mathbf{S}}$ and with V on $S \backslash \underline{\mathbf{S}}$. We now state Theorem 2.

Theorem 2 *Let $(e', V') \in \mathcal{R}_\pi$ be given. The economy $\mathcal{E}' = \{(I, S, L, J), \widetilde{V}', (S_i)_{i\in I}, (e'_i)_{i\in I}, (u_i)_{i\in I}\}$, as defined in Section 2 from above, admits an equilibrium along Definition 1.*

Proof Let $(e', V') \in \mathcal{R}_\pi$ be given. We set one $(x, y, p, L, e', V') \in \pi^{-1}(V', e') \neq \varnothing$ and let $\widetilde{V}'$ be the $(S \times L') \times J$ matrix defined as above. At no cost, we may assume that $V'(p) = L \in \mathcal{G}$. Then, from Claim 2, there exists $(z_i) \in \mathbb{R}^{J\times I}$, such that: $\sum_{i\in I} z_i = 0$ and $p_s \cdot (x_{is} - e'_{is}) = \widetilde{V}'(s, p_s) \cdot z_i$, for each $(i, s) \in I \times S_i$. From Claim 1, we let $q \in \mathbb{R}^J$ and, for each $i \in I$, $\lambda_i := (\lambda_{is}) \in \mathbb{R}^{S_i}_{++}$ be such that $q = \sum_{s\in S_i} \lambda_{is} \widetilde{V}'(s, p_s)$. It results from Claim 2 and the definitions that, for each $i \in I$, $(x_i, z_i) \in X_i \times \mathbb{R}^J$ belongs to the set:

$$B_i(p, q) := \{ \ (x, z) \in X_i \times \mathbb{R}^J : p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot z \ \ \textit{and} \ \ p_s \cdot (x_s - e_{is}) \leqslant \widetilde{V}'(s, p_s) \cdot z, \ \forall s \in S_i \ \}.$$

By construction, for each $i \in I$, $B_i^*(p,q) := \{x \in X_i : \exists z \in \mathbb{R}^J,\ (x,z) \in B_i(p,q)\}$ is included in the pseudo-equilibrium budget set. Since, for each $i \in I$, x_i is optimal in the latter set, and $x_i \in B_i^*(p,q)$, x_i is also optimal in $B_i^*(p,q)$, that is, Condition (a) of Definition 1 holds. From Claim 2 and Condition (d) of Definition 2, the stategies (x_i, z_i), for $i \in I$, also meet Conditions (b) and (c) of Definition 1, that is, $(p, q, (x_i), (z_i))$ defines an equilibrium of the economy $\mathcal{E}' = \{(I, S, L, J), \widetilde{V}', (S_i)_{i\in I}, (e'_i)_{i\in I}, (u_i)_{i\in I}\}$. $\square$

We notice that the result of Theorem 2 does not depend on assets' payoffs in unrealizable states ($s \in S \backslash \underline{\mathbf{S}}$). This theorem proves that, generically in agents' endowments and in payoffs in realizable states ($s \in \underline{\mathbf{S}}$), equilibria exist for every financial structure, where agents may have asymmetric information and anticipate normalized prices on each spot market. In a companion paper, this theorem serves to restore the full existence property of sequential equilibrium for every financial structure, by dropping the rational expectation assumptions of Radner (1972-1979).

References

[1] Cass, D., Competitive equilibrium with incomplete financial markets, CARESS Working Paper 84-09, University of Pennsylvania, 1984.

[2] Cornet, B., De Boisdeffre, L., Arbitrage and price revelation with asymmetric information and incomplete markets, J. Math. Econ. 38, 393-410, 2002.

[3] De Boisdeffre, L., No-arbitrage equilibria with differential information: an existence proof, Economic Theory 31, 255-269, 2007.

[4] De Boisdeffre, L., Learning from arbitrage, Econ Theory Bull 4, 111-119, 2016.

[5] Debreu, G., Economies with a finite set of equilibria, Econometrica 38, 387-392.

[6] Debreu, G., Smooth preferences, Econometrica 40, 603-615, 1972.

[7] Debreu, G., Smooth preferences, a corrigendum, Econometrica 44, 831-832, 1976.

[8] Duffie, D., Shafer, W., Equilibrium in incomplete markets, A basic Model of Generic Existence, J. Math. Econ. 14, 285-300, 1985.

[9] Hart, O., On the optimality of equilibrium when the market structure is incomplete, JET 11, 418-433, 1975.

[10] Mc Manus, D., Incomplete markets: Generic existence of equilibrium and optimality properties in an economy with futures markets, Upenn. Philadelphia, 1984.

[11] Magill, M., Shafer, W., Allocation of aggregate and individual risks through futures and insurance markets, University of Southern California, Dec., 1984.

[12] Magill, M., Shafer, W., Equilibrium and efficiency in a canonical asset market model, University of Southern California, 1985.

[13] Milnor, J., Topology from the Differentiable Viewpoint, Princeton Landmarks in Mathematics, Princeton, New Jersey, 1997.

[14] Repullo, R., Equilibrium and efficiency in economies with a sequence of markets, PhD. dissertation, University of London, 1984.

[15] Radner, R., Existence of equilibrium plans, prices and price expectations in a sequence of markets. Econometrica 40, 289-303, 1972.

[16] Radner, R., Rational expectations equilibrium: generic existence and the information revealed by prices. Econometrica 47, 655-678, 1979.

[17] Radner, R., Equilibrium under uncertainty, in Handbook of Math. Econ., K.J Arrrow & M.D. Intriligator Eds, North Holland, 923-1002, 1982.