Learning from arbitrage

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Abstract

We extend the refinement of information process presented in [3] to a model with uncountably many states of nature. This setting has the larger scope. It encompasses, in particular, the model of [3], where agents may have private information, and the model of [5], where they have private information, anticipations and beliefs. With no price model a la Radner (1972, 1979), and even no price to be observed, we show how agents may always infer information from financial markets, whenever required, and narrow down their anticipation sets, until all arbitrage is precluded.

Key words: anticipations, inferences, perfect foresight, rational expectations, financial markets, asymmetric information, arbitrage.

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1 Introduction

In [3], we showed that agents, exchanging assets in a financial economy with incomplete markets and asymmetric information, were still able to learn about their partners' private information when they had no price model a la Radner (1979), that is, no expectation of how equilibrium prices were determined. They inferred information by eliminating, in successive steps, their arbitrage states, that is, the states of nature that would grant them an unlimited arbitrage opportunity, if they were realizable. This model was finite dimensional and relied on the standard assumption that agents were all endowed with the perfect foresight of future prices.

In the current paper, we drop the perfect foresight restriction, and consider a model with uncountably many states (sometimes called anticipations or forecasts), a sub-set of which represents each agent's private expectations. The interpretation of the state space may be large, for it embeds the random states, upon which nature plays, but may also embed the endogenous uncertainty, stemming from agents' private actions, characteristics or beliefs. That they be private would typically result in an additional endogenous uncertainty about future prices, as shown in [5]. Thus, the current model encompasses that of [3], as a particular application case, but also that of [5], where agents have private information, anticipations and beliefs.

Agents may refine their information, that is, narrow down their sets of expected states (forecasts), in two ways. Either, they may observe a so-called "no-arbitrage price" on financial markets and infer information from that price in a decentralized manner. Or, when this is not the case, they may always infer information from mutually beneficial trade opportunities on markets. Typically, a trade-house, or financial intermediary, e.g. by seeking to make profit, would help reveal these exchange opportunities. In both cases, agents narrow down their expectation sets in finitely many steps, by elinating forecasts, that would grant them an unlimited arbitrage opportunity, if correct. It is a similar inference path as that of [3].

In Section 2, we present the basic model and concepts. In Section 3, we present no-arbitrage prices and the decentralized inferences they permit, when they may be observed. In Section 4, we introduce the coarsest arbitrage-free refinement of agents' prior information and the inferences towards that refinement, using no price.

2 The basic model

We consider a pure-exchange economy with two periods ($t \in \{0, 1\}$), where finitely many agents, $i \in I := \{1, ..., J\}$, may have private information and beliefs regarding future states, denoted by ω, which belong to a state space, denoted by Ω. Throughout, we shall take $\Omega :=]0, 1[$, which may stand for any (relatively) open subset with cardinality of the continuum of a metric space. We we will always denote by ω_0 the unique (certain) state of the first period ($t = 0$).

2.1 Information and beliefs

At $t = 0$, each agent, $i \in I$, has a private information signal represented by a closed sub-set, Ω_i, of Ω, which correctly informs her that tomorrow's state will belong to Ω_i. This set represents what the agent knows or expects to be possible tomorrow. It is, therefore, called her information (or anticipation) set. In a model with spot markets, no price model a la Radner and private anticipations, the set Ω_i should embed, in particular, all prices the agents expects to be possible tomorrow, in any state, as specified in [5]. Her assessment of the likelihood of states is, then, represented by a probabilty distribution on $(\Omega, \mathcal{B}(\Omega))$, called her belief, whose support is Ω_i ($\mathcal{B}(\Omega)$ denotes the Borel sigma-algebra of Ω).

The initial information in the economy is, thus, a typically asymmetric collection of sets, (Ω_i), that are set as given throughout the paper. Their intersection is non-empty, since agents are correctly informed and all expect tomorrow's true state as a possibility. Starting from (Ω_i), agents may narrow down their information sets and update their beliefs, along the following Definition.

Definition 1 *A collection,* $(P_i) := (P_i)_{i \in I}$ *of closed subsets of* Ω *is said to be an anticipation structure, or structure, if:*

(a) $\cap_{i=1}^m P_i \neq \varnothing$.

Their set is denoted by $\mathcal{AS}$. *A structure,* $(P'_i) \in \mathcal{AS}$, *is said to refine, or to be a refinement of* $(P_i) \in \mathcal{AS}$, *and we denote it by* $(P'_i) \leq (P_i)$, *if:*

(b) $P'_i \subset P_i$, $\forall i \in I$.

A refinement, $(P'_i) \in \mathcal{AS}$, *of* $(P_i) \in \mathcal{AS}$, *is said to be self-attainable if:*

(c) $\cap_{i=1}^m P'_i = \cap_{i=1}^m P_i$.

For every $\varepsilon > 0$, *every* $\overline{\omega} \in \Omega$ *and every probability distribution,* π, *on* $(\Omega, B(\Omega))$, *we let* $B(\omega, \varepsilon) := \{\omega \in \Omega : |\omega - \overline{\omega}| < \varepsilon\}$, *and* $P(\pi) := \{\omega \in \Omega : \pi(B(\omega, \varepsilon)) > 0, \forall \varepsilon > 0\}$ *be the support of* π. *The m probabilities,* (π_i), *on* $(\Omega, B(\Omega))$, *are said to be a structure of beliefs if* $(P(\pi_i))$ *is an anticipation structure. Then,* (π_i) *is said to support* $(P(\pi_i)) \in \mathcal{AS}$. *Given* $(P_i) \in \mathcal{AS}$, *the set of stuctures of beliefs, which support* (P_i), *is denoted by* $\Pi[(P_i)]$.

Remark 1 The above specification of information embeds that of [3], where agents' information sets are all finite. It also embeds the specification of [5], where anticipation sets are all closed subsets of $\{1, ..., K\} \times \mathbb{R}^L_{++}$, for some integers K and L, and, therefore, have the characteristics stated above.

Therefore, the current model embeds those of [3] and [5], and all its results, presented hereafter, hold in the latter models. To see this, it would suffice to complete the specification with spot markets and consumers' preferences and behaviors as in [3] or [5]. Yet, such devices need not be introduced here, in the general model, because our purpose is only to show how agents may refine their information with no price model a la Radner. In this setting, commodity prices and markets reveal no information to agents and, therefore, need not be specified for our purpose.

2.2 The asset market

Agents exchange finitely many assets, $j \in \mathcal{J} := \{1, ..., J\}$, at $t = 0$. Assets pay off at $t = 1$, in each state $\omega \in \Omega$, conditionnally on the occurence of that state. The cash payoffs, $v_j(\omega) \in \mathbb{R}$, of all assets, $j \in \mathcal{J}$, conditional on the occurence of state ω, define a row vector, $V(\omega) = (v_j(\omega)) \in \mathbb{R}^J$, whose mapping $\omega \in \Omega \mapsto V(\omega)$ is assumed to be continuous. Agents take unrestrained positions, in each asset, which are the components of her portfolio, $z \in \mathbb{R}^J$. Given an asset price, $q \in \mathbb{R}^J$, a portofolio, $z \in \mathbb{R}^J$, is thus a contract, which costs $q \cdot z$ units of account at $t = 0$, and promises to pay $V(\omega) \cdot z$ units tomorrow, in each state, $\omega \in \mathcal{M}$, if ω obtains.

3 Decentralized inferences from no-arbitrage prices

We start with a Definition.

Definition 2 *A price, $q \in \mathbb{R}^J$ is said to be a common no-arbitrage price of a structure, $(P_i) \in \mathcal{AS}$, or the structure (P_i) to be q-arbitrage-free, if the following condition holds:*
(a) $\nexists (i,z) \in I \times \mathbb{R}^J : -q \cdot z \geqslant 0$ and $V(\omega) \cdot z \geqslant 0$, $\forall \omega \in P_i$, with one strict inequality;
We denote by $Q_c[(P_i)]$ the set of common no-arbitrage prices of a given structure

$(P_i) \in \mathcal{AS}$. *A structure,* $(P_i) \in \mathcal{AS}$*, is said to be arbitrage-free if* $Q_c[(P_i)]$ *is non-empty. We say that* q *is a no-arbitrage price (respectively, a self-attainable no-arbitrage price) of a structure,* $(P_i) \in \mathcal{AS}$*, and denote it by* $q \in Q[(P_i)]$*, if there exists a refinement (resp. a self-attainable refinement),* (P_i^*)*, of* (P_i)*, such that* $q \in Q_c[(P_i^*)]$*.*

We notice that the symmetric refinement, (P_i^*), of any struture $(P_i) \in \mathcal{S}$, that is, $(P_i^*) \leq (P_i)$, such that $P_j^* = \cap_{i=1}^m P_i$ for every $j \in I$, is self-attainable and arbitrage-free.

No-arbitrage prices convey information, as stated in Claim 1, below. To show this, we set as given a price, $q \in Q[(\Omega_i)]$, and define by induction, on $n \in \mathbb{N}$, two set sequences, $\{A_i^n\}_{n\in\mathbb{N}}$ and $\{P_i^n\}_{n\in\mathbb{N}}$, for each $i \in I$, as follows:

- for $n = 1$, we let $A_i^1 = \varnothing$ and $\Omega_i^1 := \Omega_i$;
- for $n \in \mathbb{N}$ arbitrary, with A_i^n and Ω_i^n defined at step n, we let

$A_i^{n+1} := \{\overline{\omega} \in \Omega_i^n : \exists z \in \mathbb{R}^J,\ -q \cdot z \geqslant 0,\ V(\overline{\omega}) \cdot z > 0 \textit{ and } V(\omega) \cdot z \geqslant 0,\ \forall \omega \in \Omega_i^n\}$;

$\Omega_i^{n+1} := \Omega_i^n \setminus A_i^{n+1}$, i.e., the agent rules out expected states, granting an arbitrage.

Claim 1 *Let a no-arbitrage price,* $q \in Q[(\Omega_i)]$*, and the above defined sesquences,* $\{A_i^n\}_{n\in\mathbb{N}}$ *and* $\{\Omega_i^n\}_{n\in\mathbb{N}}$*, be given. Then, the following Assertions hold:*
(i) *there exists a coarsest q-arbitrage free refinement of* (Ω_i)*, denoted by* $(\Omega_i(q))$*, in the sense that* $(\Omega_i(q))$ *is q-arbitrage-free and every q-arbitrage-free refinement of* (Ω_i) *refines* $(\Omega_i(q))$*. Moreover, if* $q \in Q[(\Omega_i)]$ *is self-attainable,* $(\Omega_i(q))$ *is self-attainable.*
(ii) $\exists N \in \mathbb{N} :\ \forall n > N, \forall i \in I,\ A_i^n = \varnothing$ *and* $\Omega_i^n = \Omega_i(q)$*.*

Proof The proof results directly, mutatis mutandis, from Claims 2, 3 & 4 of [5].

Along Claim 1, if $(\Omega_i) \in \mathcal{AS}$ is arbitrage-free at the outset, agents would not refine (nor need refine) their information before reaching agreement on a price assessment of assets. If it is not so and if, for some reason, assets may be traded at a no-arbitrage price on markets, then, all agents, although unaware of how market prices are determined, would infer information from observing that price, until all arbitrage vanished. This seems to be what actually happens on the stock exchange. Financial intermediaries would take advantage of fictitious arbitrage opportunities percieved by traders having incomplete information, up to the point where the latter agents have narrowed down their anticipation sets to an arbitrage-free structure. Yet, the question arises why assets are exchanged at a single (no-arbitrage) price, when the anticipation structure is not arbitrage-free, and, therefore, prevents any agreement on the assessment of asset prices. Hereafter, we propose a solution to that problem.

4 A refinement path through trade

4.1 Characterizing no-arbitrage

Claim 2 characterizes common no-arbitrage prices and structures.

Claim 2 *Let* $(P_i) \in \mathcal{AS}$*,* $(\pi_i) \in \Pi[(P_i)]$ *and* $q \in \mathbb{R}^J$ *be given, along Definition 1. For each* $i \in I$*, we denote by* $L_2^+(\pi_i)$ *and* $L_2^{++}(\pi_i)$*, respectively, the sub-sets of mappings,* $f : P_i \to \mathbb{R}$*, in the Riesz space* $L_2(\pi_i)$*, such that* $f(\omega) \geqslant 0$ *and* $f(\omega) > 0$ π_i*-almost surely*[2]*. Then, the following statements are equivalent:*

(i) $q \in Q_c[(P_i)]$, that is, (P_i) is q-arbitrage free;

(ii) $\forall i \in I$*,* $\exists f_i \in L_2^{++}(\pi_i)$*, such that* $q = \int_{\omega \in P_i} V(\omega) f_i(\omega) d\pi_i(\omega)$;

Moreover, (P_i) *is arbitrage-free if and only if it meets the following AFAO Condition:*

[2] For the sake of clarity, $L_2^{++}(\pi_i)$ is the sub-set of mappings $f : P_i \to \mathbb{R}$, in $L_2(\pi_i)$, such that, for every $\overline{\omega} \in P_i$, every $\varepsilon > 0$, and $B = \{\omega \in P_i : \|\omega - \overline{\omega}\| < \varepsilon\}$, the following relation holds: $\int_{\omega \in B} f(\omega) d\pi_i(\omega) > 0$.

There is no portfolio collection $(z_i) \in (\mathbb{R}^J)^I$*, such that* $\sum_{i=1}^m z_i = 0$ *and* $V(\omega_i) \cdot z_i \geqslant 0$ *for every pair* $(i, \omega_i) \in I \times P_i$*, with at least one strict inequality.*

Proof We set $(P_i) \in \mathcal{AS}$ and $q \in \mathbb{R}^J$ as given and use the notations of Claim 2.

$(ii) \Rightarrow (i)$ Assume that Assertion (ii) holds and let $i \in I$ be given and $f_i \in L_2^{++}(\pi_i)$ be such that $q = \int_{\omega \in P_i} V(\omega) f_i(\omega) d\pi_i(\omega)$. Let $z \in \mathbb{R}^J$ be such that $-q \cdot z \geqslant 0$ and $V(\omega) \cdot z \geqslant 0$ for every $\omega \in P_i$. Assume, first, that $V(\overline{\omega}) \cdot z > 0$, for some $\overline{\omega} \in P_i$. Then, the above inequalities $V(\omega) \cdot z \geqslant 0$, which hold for every $\omega \in P_i$, and the continuity of V at $\overline{\omega}$ imply $q \cdot z = \int_{\omega \in P_i} V(\omega) \cdot z f_i(\omega) d\pi_i(\omega) > 0$, contradicting the above relation $-q \cdot z \geqslant 0$. Hence, $V(\omega) \cdot z = 0$, for all $\omega \in P_i$ and $q \cdot z = 0$, and Assertion (i) of Claim 2 holds. □

$(i) \Rightarrow (ii)$ Assume that Assertion (i) holds and let $i \in I$ and $P'_i := \{\omega_0\} \cup P_i$ be given and L_i be the set of mappings from P'_i to $\mathbb{R}$, whose restriction to P_i is in the Riesz space $L_2(\pi_i)$, endowed with the duality $(f, g) \in L_i^2 \mapsto < f, g > := f(\omega_0) g(\omega_0) + \int_{\omega \in P_i} f(\omega) g(\omega) d\pi_i(\omega)$, norm $f \in L_i \mapsto \|f\| := \sqrt{f(\omega_0)^2 + \int_{\omega \in P_i} f(\omega)^2 d\pi_i(\omega)}$ and metric topology. Thus, L_i is a convex metric space, with linear sub-spaces:

$$A_i := \{f \in L_i : \exists z \in \mathbb{R}^J,\ f(\omega_0) = -q \cdot z \text{ and } f(\omega) = V(\omega) \cdot z,\ \forall \omega \in P_i\};$$
$$A_i^{\perp} := \{f \in L_i : < a, f > = 0,\ \forall a \in A\}.$$

Let L_i^+ (respectively, L_i^{++}) be the subsets of mappings, $f : P'_i \to \mathbb{R}$, in L_i, such that $f(\omega_0) \geqslant 0$ (resp., $f(\omega_0) > 0$), and whose restriction to P_i belongs to $L_2^+(\pi_i)$ (resp., to $L_2^{++}(\pi_i)$). Assertion (i) is written $A_i \cap L_i^+ = \{0\}$. Assume, by contraposition, that $A_i^{\perp} \cap L_i^{++} = \varnothing$, i.e., Assertion (ii) fails (which implies that $\omega \in P_i \mapsto V(\omega)$ is nonzero).

From Assertion (i) and above, the nonempty cone $L_i^{++} - A_i^{\perp}$ is not dense (e.g., the mapping $g \in L_i$, defined by $g(\omega) = -1$, for every $\omega \in P'_i$, is not in the closure of the cone $L_i^{++} - A_i^{\perp}$, which is $L_i^+ - A_i^{\perp}$). Hence, from ([1], Lemmas 5.44, p.188, and

5.74, p. 203) there exists a nonzero continuous linear functional, φ, which separates $A_i^{\perp}$ and L_i^{++}, such that: $\varphi(a) = 0 \leqslant \varphi(b)$, for every $(a, b) \in A_i^{\perp} \times L_i^{++}$.

From the Riesz' representation (see [1], pp. 208, 440), there exists $f_i \in L_i$, such that $\varphi(h) = < f_i, h >$, for every $h \in L_i$. The linear space A_i is closed and finite dimensional, hence, with an obvious definition, $A_i^{\perp\perp} = A_i$ (see [1], p. 215). Then, from the above inequalities, the relations $f_i \in A_i^{\perp\perp} \cap L_i^{+} \backslash \{0\} = A_i \cap L_i^{+} \backslash \{0\}$ hold and contradict the above restatement, $A \cap\ L_i^{+} = \{0\}$, of Assertion (i). □

The fact that (P_i) meets the AFAO Condition if it is arbitrage-free is proved, mutatis mutandis, in [5]. □

We now assume that (P_i) meets the AFAO Condition. For each $i \in I$, we define L_i, L_i^{+} and L_i^{++} as above and let $\mathcal{L} := \times_{i \in I} L_i$, $\mathcal{L}^{+} := \times_{i \in I} L_i^{+}$ and $\mathcal{L}^{++} := \times_{i \in I} L_i^{++}$ be endowed with the operator, metric and topology of product spaces, and let:

$$A := \{(f_i) \in \mathcal{L} : (f_i(\omega_0)) = 0,\ \exists (z_i) \in \mathbb{R}^{JI} : \textstyle\sum_{i=1}^{m} z_i = 0,\ f_i(\omega_i) = V(\omega_i) \cdot z_i,\ \forall (i, \omega_i) \in I \times P_i\};$$

$$A^{\perp} := \{f \in \mathcal{L} : < a, f > = 0, \forall a \in A\}.$$

The AFAO Condition is written: $A \cap \mathcal{L}^{+} = \{0\}$. If we had $A^{\perp} \cap\ \mathcal{L}^{++} = \varnothing$, the very same arguments as above would apply and (as we let the reader check) yield a contradiction. Hence, we set as given $(f_i) \in A^{\perp} \cap\ \mathcal{L}^{++} \neq \varnothing$. By taking $(z_i) \in (\mathbb{R}^{J})^{I}$, such that $(z_i, z_j) = (-z_1, 0)$, for every $(i, j) \in I^2$, $i \neq 1$, $j \notin \{1, i\}$, the relation $(f_i) \in A^{\perp}$ yields: $\int_{\omega \in P_i} f_i(\omega) V(\omega) \cdot z d\pi_i(\omega) = \int_{\omega \in P_1} f_1(\omega) V(\omega) \cdot z d\pi_1(\omega)$, for every pair $(i, z) \in I \times \mathbb{R}^{J}$. Then, if we let $q := \int_{\omega \in P_1} f_1(\omega) V(\omega) d\pi_1(\omega)$, it follows from above that $q = \int_{\omega \in P_i} f_i(\omega) V(\omega) d\pi_i(\omega)$, for each $i \in I$, and, from Assertion (ii) and above, that (P_i) is arbitrage-free. □

4.2 The coarsest arbitrage-free refinement

We show the initial information, (Ω_i), admits a coarsest arbitrage-free refinement.

Claim 3 *The structure,* $(\Omega_i) \in \mathcal{AS}$*, admits a coarsest arbitrage-free refinement, which is unique and self-attainable, namely, a refinement,* $(\Omega_i^*) \leq (\Omega_i)$*, such that:*

(i) (Ω_i^*) *is arbitrage-free;*

(ii) every arbitrage-free refinement of (Ω_i) *refines* (Ω_i^*).

Proof Let $\mathcal{R}$ be the set of arbitrage-free refinements of (Ω_i). That set contains the symmetric self-attainable refinement of (Ω_i). Let $\Omega_i^* = \overline{\cup_{(P_i)\in\mathcal{R}} P_i}$, for every $i \in I$. By construction, $(\Omega_i^*) \leq (\Omega_i)$ is self-attainable and satisfies assertion (ii) of Claim 3. Assume, by contraposition, that (Ω_i^*) is not arbitrage-free, that is, from Claim 2, there exist portfolios $(z_i) \in (\mathbb{R}^J)^I$, such that $\sum_{i=1}^m z_i = 0$ and $V(\omega_i) \cdot z_i \geqslant 0$ for every couple $(i, \omega_i) \in I \times \Omega_i^*$, with at least one strict inequality, say, for $i = 1$ and $\overline{\omega} \in \Omega_1^*$. From the continuity of $\omega \mapsto V(\omega)$, and the definition of (Ω_i^*), there exists $(P_i) \in \mathcal{R}$ and $\overline{\omega}_1 \in P_1$, close enough to $\overline{\omega}$, such that, $\sum_{i=1}^m z_i = 0$, $V(\omega_i) \cdot z_i \geqslant 0$ for every couple $(i, \omega_i) \in I \times P_i$ and $V(\overline{\omega}_1) \cdot z_1 > 0$, which (from Claim 2) contradicts the fact that (P_i) is arbitrage-free. This contradiction proves that (Ω_i^*) also meets Assertion (i). □

We now show how agents may infer the above refinement, (Ω_i^*), from the market.

4.3 Sequential refinement through trade

Henceforth, we assume agents' initial information, $(\Omega_i) \in \mathcal{AS}$, yields an arbitrage. As long as it lasts, agents cannot agree on a price assessment of assets. Yet, they may narrow down in steps their information sets from observing exchange opportunities on financial markets. To see this, we define, by induction on $n \in \mathbb{N}$, the following sequences, $\{(A_i^n)\}_{n\in\mathbb{N}}$ and $\{(\Omega_i^n)\}_{n\in\mathbb{N}}$, of sub-sets of $(\{\varnothing\} \cup \Omega)^m$:

- we let $A_i^1 = \varnothing$ and $\Omega_i^1 := \Omega_i$, for each $i \in I$;
- with A_i^n and P_i^n defined at step $n \in \mathbb{N}$, for each $i \in I$, we let, for each $i' \in I$:

$$A_{i'}^{n+1} := \{\overline{\omega} \in \Omega_{i'}^{n} : \exists (z_i) \in (\mathbb{R}^J)^m,\ \sum_{i=1}^{m} z_i{=}0,\ V(\overline{\omega}){\cdot}z_{i'}{>}0,\ V(\omega_i){\cdot}z_i{\geqslant}0,\ \forall (i,\omega_i) \in I{\times}\Omega_i^n\}$$

$$\Omega_{i'}^{n+1} := \Omega_{i'}^{n} \setminus A_{i'}^{n+1}$$

In the above refinement steps, agents rule out expectations, granting an arbitrage, because they may eventually trust the market over their incomplete information and realize that what they initially thought to be an arbitrage was fictitious. As mentionned above, it seems that the stock exchange operates this way, with financial intermediaries taking advantage of agents' incomplete information, and selling profitable zero-sum portfolio bundles, as long as they can. However, as time elapses and competition takes place, these portfolios' prices tend to zero and agents eventually infer that their - once perceived - arbitrage opportunities were fictitious ones. They would refine their information accordingly. This refinement path would lead them to infer the above arbitrage-free structure, (Ω_i^*), as shown by Claim 4.

Claim 4 *Let $(\Omega_i^*) \in \mathcal{AS}$ be the coarsest arbitrage-free refinement of agents' prior information, $(\Omega_i) \in \mathcal{AS}$. Let $\{(A^n)\}_{n\in\mathbb{N}}$ and $\{(\Omega_i^n)\}_{n\in\mathbb{N}}$, be defined as above. The following Assertions hold:*

(i) $\exists N \in \mathbb{N} : \forall n > N, \forall i \in I,\ A_i^n = \varnothing$ and $\Omega_i^n = \Omega_i^N$;

(ii) $(\Omega_i^N) = (\Omega_i^*)$, along Assertion (i).

Proof Let $\{(A_i^n)\}_{n\in\mathbb{N}}$ and $\{(\Omega_i^n)\}_{n\in\mathbb{N}}$ be defined as above and $(\Omega_i^{**}) := \lim\searrow(\Omega_i^n)$.

First, we show that the relations $(\Omega_i^*) \leq (\Omega_i^n) \leq (\Omega_i)$ hold for every $n \in \mathbb{N}$. They hold from the definition and Claim 3 for $n = 1$, since $(\Omega_i^1) := (\Omega_i)$. Assume that $(\Omega_i) \leq (\Omega_i^n) \leq (\Omega_i)$ holds for a given integer, $n \in \mathbb{N}$. Then, for each $i \in I$, Ω_i^n is closed, and so is Ω_i^{n+1} from the definition and the continuity of $\omega \mapsto V(\omega)$. Assume, by contraposition, that there exists $i \in I$, say $i = 1$, such that $\Omega_1^* \subset \Omega_1^n$ and $\Omega_1^* \not\subseteq \Omega_1^{n+1}$. Then, from the definitions, there exist $\overline{\omega} \in\ \Omega_1^*\ \cap\ A_1^{n+1}$ and $(z_i) \in (\mathbb{R}^J)^m$, such that

$\sum_{i'=1}^{m} z_i = 0$, $V(\overline{\omega})\cdot z_1 > 0$ and $V(\omega_i)\cdot z_i \geqslant 0$, for every $(i,\omega_i) \in I\times\Omega_i^* \subset I\times\Omega_i^n$, which contradicts Claims 2 & 3, along which (Ω_i^*) meets the AFAO Condition.

Hence, the relations $(\Omega_i^*) \leq (\Omega_i^n) \leq (\Omega_i)$ hold for all $n \in \mathbb{N}$, which implies, passing to the limits on nonempty intersections of compact sets: $(\Omega_i^*) \leq (\Omega_i^{**}) \leq (\Omega_i)$.

For each $i \in I$, let $Z_i^{on} := \{z \in \mathbb{R}^J : V(\omega)\cdot z = 0,\ \forall\omega \in \Omega_i^n\}$. Since $\{(\Omega_i^n)\}_{n\in\mathbb{N}}$ is non-increasing, the sequence of vector spaces, $\{\times_{i\in I} Z_i^{on}\}$, is non-decreasing in $(\mathbb{R}^J)^m$, hence, stationary. We let $N \in \mathbb{N}$ be such that $\times_{i\in I} Z_i^{on} = \times_{i\in I} Z_i^{oN}$, for every $n \geqslant N$. Assume, by contraposition, that assertion (i) of Claim 4 fails, that is:

$$\forall n \in \mathbb{N}, \exists(\omega_{i_n}^n, (z_i^n)) \in \Omega_{i_n}^n \times \mathbb{R}^{Jm} : \sum_{i=1}^{m} z_i^n = 0,\ V(\omega_{i_n}^n)\cdot z_{i_n}^n > 0 \textit{ and } V(\omega_i)\cdot z_i^n \geqslant 0, \forall(i,\omega_i) \in I\times\Omega_i^n.$$

From the definition of (Ω_i^n) and (Ω_i^{n+1}), the above portfolios satisfy, for all $n \in \mathbb{N}$, $(z_i^n) \notin \times_{i\in I} Z_i^{on}$ and $(z_i^n) \in \times_{i\in I} Z_i^{o(n+1)}$, which is impossible, from above, if $n \geqslant N$. This contradiction proves Assertion (i) of Claim 4, for the integer $N \in \mathbb{N}$ introduced above. Moreover, $(\Omega_i^{**}) = (\Omega_i^N)$, is q-arbitrage-free (since $A_i^{N+1} = \varnothing$, for each $i \in I$), which yields, from Claim 3 and above: $(\Omega_i^{**}) \leq (\Omega_i^*) \leq (\Omega_i^{**}) \leq (\Omega_i)$. That is, $(\Omega_i^{**}) = (\Omega_i^*) = (\Omega_i^N)$, and assertion (ii) of Claim 4 also holds. This completes the proof. □

Thus, agents may always refine their information with no price (nor price model) and reach an arbitrage-free anticipation structure. If we apply this result to the model of [5], agents having inferred (Ω_i^*) will always be able to reach equilibrium, if their initial structure embeds the so-called ‘minimum uncertainty set’. This set represents the incompressible uncertainty in the economy resulting from the fact that agents’ beliefs are private (see Theorem 1 of [5]). Then, the structure (Ω_i^*), inferred with no price, cannot be refined any further. As shown by Theorem 1 in [5], any structure of beliefs, $(\pi_i) \in \Pi(\Omega_i^*)$, is consistent with equilibrium, but equilibrium prices convey no information. They might change with agents beliefs, but always reveal the same structure, (Ω_i^*). Thus, in our model, the path to equilibrium discards rational expectations, that is, a joint determination of equilibrium prices and anticipations, relying on expectations a la Radner. Agents' inferences from markets use no price model, but only arbitrage, as seems to be the case on actual markets.

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