Financial Equilibrium with differential Information: a Theorem of generic Existence

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Abstract

We propose a proof of generic existence of equilibrium in a pure exchange economy, where agents are typically asymmetrically informed, exchange commodities, on spot markets, and securities of all kinds, on incomplete financial markets. The proof does not use Grasmanians, nor differential topology (except Sard's theorem), but good algebraic properties of assets' payoffs, whose spans, generically, never collapse. Then, we show that an economy, where the payoff span cannot fall, admits an equilibrium. As a corollary, we prove the full existence of financial equilibrium for numeraire assets, extending Geanakoplos-Polemarchakis (1986) to the asymmetric information setting. The paper, which still retains Radner's (1972) standard perfect foresight assumption, is also a milestone to prove, in a companion article, the existence of sequential equilibrium when the classical rational expectation assumptions, along Radner (1972, 1979), are dropped jointly, that is, when agents have private characteristics and beliefs and no model to forecast prices.

Key words: sequential equilibrium, temporary equilibrium, perfect foresight, existence, rational expectations, financial markets, asymmetric information, arbitrage.

JEL Classification: D52

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1 Introduction

This paper proposes a non standard proof of the generic existence of equilibrium in incomplete financial markets with differential information. It presents a two-period pure exchange economy, with an ex ante uncertainty over the state of nature to be revealed at the second period. Asymmetric information is represented by private finite subsets of states, which each agent is correctly informed to contain the realizable states. Finitely many consumers exchange consumption goods on spot markets, and, unrestrictively, assets of any kind on incomplete financial markets. Those permit limited transfers across periods and states. Agents have endowments in each state, preferences over consumptions, possibly non ordered, and no model to forecast prices. Generalizing Cass (1984) to asymmetric information, De Boisdeffre (2007) shows the existence of equilibrium on purely financial markets is characterized, in this setting, by the absence of arbitrage opportunities on financial markets, a condition which can be achieved with no price model, along De Boisdeffre (2016), from simply observing available transfers on financial markets.

When assets pay off in goods, equilibrium needs not exist, as shown by Hart (1975) in the symmetric information case. His example is based on the collapse of the span of assets' payoffs, that occurs at clearing prices. In our model, an additional problem arises from differential information. Financial markets may be arbitrage-free for some commodity prices, and not for others, in which case equilibrium cannot exist. We show this problem vanishes owing to a good property of payoff matrixes.

Attempts to resurrect the existence of equilibrium with real assets noticed that the above "*bad*" prices could only occur exceptionally, as a consequence of Sard's theorem. These attempts include Mc Manus (1984), Repullo (1984), Magill & Shafer

(1984, 1985), for potentially complete markets (i.e., complete for at least one price), and Duffie-Shafer (1985, 1986), for incomplete markets. These papers apply to symmetric information, build on differential topology arguments, and demonstrate the generic existence of equilibrium, namely, existence except for a closed set of measure zero of economies, parametrized by the assets' payoffs and agents' endowments.

The current paper proposes to show generic existence differently, under milder assumptions and for economies parametrized by assets' payoffs only (in a restricted sense). This result applies to both potentially complete or incomplete markets, to ordered or non transitive preferences, and to symmetric or asymmetric information.

The proof does not use Grassmanians, but properties of payoff matrixes and lower semicontinuous correspondences built upon them. It yields well-behaved normalized price anticipations at equilibrium, which serve to prove, in a companion paper, the full existence of sequential equilibrium in an economy where both Radner's (1972, 1979) rational expectations are dropped. That is, agents endowed with private beliefs and characteristics can no longer forecast equilibrium prices perfectly.

So the paper be self-contained, we resume some techniques of De Boisdeffre (2007). Finally, we derive the full existence of equilibria for numeraire assets as a corollary, using a different and asymptotic technique. This latter result extends Geanakoplos-Polemarchakis' (1986) to the asymmetric information setting.

The paper is organized as follows: Section 2 presents the model; Section 3 states and proves the existence Theorem; Section 4 shows the existence of equilibria for numeraire assets; an Appendix proves a technical Lemma.

2 The model

We consider a pure-exchange economy with two periods, $t \in \{0, 1\}$, and an uncertainty, at $t = 0$, on which state of nature will randomly prevail, at $t = 1$. Consumers exchange consumption goods, on spots markets, and assets of all kinds, on typically incomplete financial markets. The sets, I, S, H and J, respectively, of consumers, states of nature, consumption goods and assets are all finite. The non random state at the first period ($t = 0$) is denoted by $s = 0$ and we let $\Sigma' := \{0\} \cup \Sigma$, for every subset, Σ, of S. Similarly, $l = 0$ denotes the unit of account and we let $H' := \{0\} \cup H$.

2.1 Markets and information

Agents consume or exchange the consumption goods, $h \in H$, on both periods' spot markets. At $t = 0$, each agent, $i \in I$, receives privately some correct information signal, $S_i \subset S$ (henceforth set as given), that tomorrow's true state will be in S_i. We assume costlessly that $S = \cup_{i \in I} S_i$. Thus, the pooled information set, $\underline{\mathbf{S}} := \cap_{i \in I} S_i$, always containing the true state, is non-empty, and $\underline{\mathbf{S}} = S$ under symmetric information. A collection of $\#I$ subsets of S, whose intersection is non-empty, is called an (information) structure, which agents may possibly refine before trading.

Since no state from the set $S \backslash \underline{\mathbf{S}}$ may prevail, we assume that each agent, $i \in I$, forms an idiosyncratic anticipation, $p_i := (p_{is}) \in \mathbb{R}_{++}^{S_i \backslash \underline{\mathbf{S}}}$ of tomorrow's commodity prices in such states, if $S_i \neq \underline{\mathbf{S}}$. Yet, to alleviate subsequent definitions and notations, we will take $p_{is} = p_{js} = \overline{p}_s$ (henceforth given), for any two agents $(i, j) \in I^2$ such that $s \in S_i \cap S_j \backslash \underline{\mathbf{S}}$. This simplification does not change generality. Thus, we may restrict tomorrow's price set to $P := \{p := (p_s) \in (\mathbb{R}_+^H)^S : \|p_s\| \leqslant 1, \forall s \in \underline{\mathbf{S}},\ p_s = \overline{p}_s,\ \forall s \in S \backslash \underline{\mathbf{S}}\}$, and we refer to any pair, $\omega := (s, p_s) \in S \times \mathbb{R}^H$, as a forecast, whose set is denoted Ω.

Agents may operate transfers across states in S' by exchanging, at $t=0$, finitely many assets, $j \in J$, which pay off, at $t=1$, conditionally on the realization of forecasts, $\omega \in \Omega$. We will assume that $\#J \leqslant \#\underline{\mathbf{S}}$, so as to cover incomplete markets. These conditional payoffs may be nominal or real or a mix of both. They are identified to the continuous map, $V : \Omega \to \mathbb{R}^J$, relating forecasts, $\omega := (s,p) \in \Omega$ to rows, $V(\omega) \in \mathbb{R}^J$, of assets' cash payoffs, delivered if state s and price p obtain. Thus, for every pair $((s,p),j) \in \Omega \times J$, the j^{th} asset delivers the vector $v_j(s) := (v_j^h(s)) \in \mathbb{R}^{H'}$ of payoffs if state s prevails, namely, $v_j^0(s) \in \mathbb{R}$ units of account, and a quantity, $v_j^h(s) \in \mathbb{R}$, of each good $h \in H$, whose cash value at price p is $v_j(\omega) := v_j^0(s) + \sum_{h \in H} p^h v_j^h(s)$. For all $j \in J$, we let $V_j := (v_j(\omega))_{\omega \in \Omega}$ be the column vector of the asset's payoffs across forecasts.

For $p := (p_s) \in P$, we let $V(p)$ be the $S{\times}J$ matrix, whose generic column is denoted $V_j(p)$, and whose generic row is $V(p,s){:=}V(s,p_s)$ (for $s \in S$); we let $V_{\underline{\mathbf{S}}}(p)$ be its truncation to $\underline{\mathbf{S}}$ and $< V_{\underline{\mathbf{S}}}(p) >$ be the span of $V_{\underline{\mathbf{S}}}(p)$ in $\mathbb{R}^{\underline{\mathbf{S}}}$.

At asset price, $q \in \mathbb{R}^J$, agents may buy or sell unrestrictively portfolios, $z = (z_j) \in \mathbb{R}^J$, for $q \cdot z$ units of account at $t=0$, against the promise of delivery of a flow, $V(\omega) \cdot z$, of conditional payoffs across forecasts, $\omega \in \Omega$. The model is dispensed with the so-called "*regularity condition*" on V, otherwise used in generic existence proofs.

For notational purposes, we let $\mathcal{V}$ be the set of $(S \times H') \times J$ payoff matrixes, as defined above, and $\mathcal{M}_0$ be the set of matrixes having zero payoffs in any good and any state $s \in S \backslash \underline{\mathbf{S}}$ (i.e., assets are nominal and pay off in realizable states only). For every $\lambda > 0$, we let $\mathcal{M}_\lambda := \{M_0 \in \mathcal{M}_0,\ \|M_0\| \leqslant \lambda\}$ and $\mathcal{V}_\lambda := \{M \in \mathcal{V} : \|M\text{-}V\| \leqslant \lambda\}$. The sets $\mathcal{M}_0$ and $\mathcal{V}$ are equiped with the same notations as above (defined for $V \in \mathcal{V}$). Non restrictively along De Boisdeffre (2016), we assume that, before trading, agents have always inferred from markets the information needed to preclude arbitrage.

We point out the good algebraic properties of payoff and financial structures, summarized in the following Claim 1, which later serve to circumvent the possible fall in rank problems a la Hart (1975). Claim 1 shows that, except for a closed negligible set of assets' payoffs, no such fall in rank may occur. Then, we will show that an economy, where the payoff span can never collapse, admits an equilibrium.

Claim 1 *Given* $(\lambda, M, p, \varepsilon) \in]0,1[\times\mathcal{V}_\lambda\times P\times]0,\lambda[$, *we let* $\mathcal{V}_\lambda^p := \{ M \in \mathcal{V}_\lambda : rank(M_{\underline{\mathbf{S}}}(p)) = \#J \}$, $\Lambda_\lambda := \{M \in \mathcal{V}_\lambda : M \in \mathcal{V}_\lambda^p,\ \forall p \in P\}$ *and* $B^o(M,p,\varepsilon) := \{(M',p') \in \mathcal{V}_\lambda\times P : \|M'-M\| + \|p'-p\| < \varepsilon\}$ *and, similarly,* $B^o(p,\varepsilon) := \{p' \in P : \|p'\text{-}p\| < \varepsilon\}$ *be given. The following Assertions hold:*

(i) *if* $M \in \mathcal{V}_\lambda^p,\ \exists\ \varepsilon' > 0 : (M',p') \in B^0(M,p,\varepsilon') \Longrightarrow M' \in \mathcal{V}_\lambda^{p'}$;

(ii) *if* $M \notin \mathcal{V}_\lambda^p,\ \exists\ M_\varepsilon \in \mathcal{V}_\varepsilon^p$;

(iii) $\exists\ M_0 \in \mathcal{M}_0 : \forall (M',p') \in \mathcal{V}_\lambda \times P,\ \exists\ \mu \in \mathbb{R},\ (M' + \mu M_0) \in \mathcal{V}_\lambda^{p'}$;

(iv) *along Assertion* (iii), $\exists \mu \in \mathbb{R} : (V + \mu M_0) \in \Lambda_\lambda$;

(v) $\forall M' \in \Lambda_\lambda$, $\nexists((z_i),p) \in (\mathbb{R}^J)^I\backslash\{0\} \times P : \sum_{i\in I}\ z_i = 0$ *and* $M'(p,s_i)\cdot z_i \geqslant 0,\ \forall (i,s_i) \in I \times S_i$. *And we say that the payoff and information structure,* $[M',(S_i)]$, *is arbitrage-free;*

(vi) *the above set,* $\Lambda_\lambda := \{M \in \mathcal{V}_\lambda : M \in \mathcal{V}_\lambda^p,\ \forall p \in P\}$, *is non empty and open in* $\mathcal{V}_\lambda$*;*

(vii) *the set,* $\{M_{\underline{\mathbf{S}}}(p) : M \in \mathcal{V}_\lambda,\ p \in P,\ M \notin \mathcal{V}_\lambda^p\}$, *of* $\underline{\mathbf{S}} \times J$ *payoff matrixes, which fall in rank, is closed and negligible.*

Proof

- Assertions (i)-(ii) are well-known. Their proofs are obvious from the definitions and the continuity of the scalar product, and left to the reader. □

- Assertion (iii): let $M_0 \in \mathcal{M}_0$, with full column rank, and $(M',p') \in \mathcal{V}_\lambda\times P$ be given. To simplify notations, we assume w.l.o.g. that $\underline{\mathbf{S}} = S$. It is clear that, for $\mu > 0$ small enough, either $(M' + \mu M_0) \in \mathcal{V}_\lambda^{p'}$ or $(M' - \mu M_0) \in \mathcal{V}_\lambda^{p'}$. From Assertion (i), the latter result holds, if $M' \in \mathcal{V}_\lambda^{p'}$. If not, let $(e_k)_{1\leqslant k\leqslant K}$ be an orthonormal basis of $A := \{v \in \mathbb{R}^J : M'(p')v = 0\}$ and $(e_k)_{K<k\leqslant \#J}$ be an orthonormal basis of $A^\perp$. By construction, the systems $\{M'(p')e_k\}_{K<k\leqslant \#J}$, $\{(M'(p')+\mu M_0(p'))e_k\}_{K<k\leqslant \#J}$ and $\{(M'(p')-\mu M_0(p'))e_k\}_{K<k\leqslant \#J}$ are all linearly independent, for $\mu > 0$ small enough, so that $\{(M'(p') + \mu M_0(p'))e_k\}_{1\leqslant k\leqslant \#J}$ and $\{(M'(p') - \mu M_0(p'))e_k\}_{1\leqslant k\leqslant \#J}$ are also linearly independent by construction. Assertion (iii) follows. □

- Assertion (iv) Assume, by contraposition, that Assertion (iv) fails, namely: $\forall n \in \mathbb{N},\ \exists\ p_n \in P,\ (V + M_0/n) \notin \mathcal{V}_\lambda^{p_n}$. Then, the sequence $\{p_n\}$ may be assumed to converge in a compact set, say to $p^* \in P$. From Assertions (i)-(iii), there exist $\mu^* > 0$ and $\varepsilon^* > 0$, small enough, such that $(V + \mu^* M_0) \in \mathcal{V}_\lambda^{p'}$ for every $p' \in B^o(p^*, \varepsilon^*)$, which contains an ending section of the sequence $\{p_n\}$, say $\{p_n\}_{n\geqslant N}$. We let the reader check, as tedious but straightforward from continuity arguments, the fact that $(V + M_0/n)(p_n)$ tends to $V(p^*)$ and has same rank for big enough integers, and from the arguments of the proof of Assertion (iii), that the latter relations, $(V + \mu^* M_0) \in \mathcal{V}_\lambda^{p_n}$ and $(V + \mu^* M_0) \in \mathcal{V}_\lambda^{p^*}$, imply, for $n \geqslant N$ large enough, $(V + M_0/n) \in \mathcal{V}_\lambda^{p_n}$, in contradiction with the above. □

- Assertion (v): let $M' \in \Lambda_\lambda$ and $((z_i), p) \in (\mathbb{R}^J)^I \times P$ be given, such that $\sum_{i\in I} z_i = 0$ and $M'(p, s_i) \cdot z_i \geqslant 0$ for every $(i, s_i) \in I \times S_i$. It follows from the fact that $M'(p)$ has full rank, that the latter relations imply $(z_i) = 0$, proving Assertion (v). □

- Asssertion (vi): let $M' \in \Lambda_\lambda$, a non-empty set from Assertion (iv), be given. Assume, by contraposition: $\forall n \in \mathbb{N}, \exists (M_n, p_n) \in \mathcal{V}_\lambda \times P,\ \|M' - M_n\| < 1/n,\ M_n \notin \mathcal{V}_\lambda^{p_n}$. By the same token as above, we let $\lim_{n\to\infty} p_n = p^* \in P$. From Assertion (i), there exists $\varepsilon^* > 0$, small enough, such that $M" \in \mathcal{V}_\lambda^{p"}$ for every $(M", p") \in B^o(M', p^*, \varepsilon^*)$, which contains an ending section of $\{(M_n, p_n)\}$, say $\{(M_n, p_n)\}_{n\geqslant N}$. The latter relations imply, $M_n \in \mathcal{V}_\lambda^{p_n}$, for $n \geqslant N$, contradicting the former. □

- To simplify, we assume throughout that $S = \underline{\mathbf{S}}$ and, at first, that $\#S = \#J$. We let $f : \mathcal{V} \times (\mathbb{R}^H)^S \times \mathbb{R}^J \to \mathbb{R}^J$ be defined (with model's notations) by $f(V', p', \lambda) := \sum_{j \in J} \lambda_j V'_j(p')$, for every $(V', p', \lambda := (\lambda_j)) \in \mathcal{V} \times (\mathbb{R}^H)^S \times \mathbb{R}^J$. From Sard's theorem (see, e.g., Milnor, 1997, p. 16), let C be the set of critical points of f (i.e., such that $rank(df_{(V',p',\lambda)}) < \#J$) then, $f(C)$, called the set of singular values, has measure zero. If $\#J < \#S$, we apply the same arguments as above to any subset $T \subset S$ of $\#J$ states and the corresponding truncated prices and $J \times J$ matrixes. The union of all singular values so obtained, hence, also the closed (from Assertion (vi)) set $\{M(p) : M \in \mathcal{V}_\lambda,\ p \in P,\ M \notin \mathcal{V}^p_\lambda\}$, have measure zero. □

2.2 The agent's behaviour and the concept of equilibrium

Each agent, $i \in I$, receives an endowment, $e_i := (e_{is})$, granting the conditional commodity bundles, $e_{i0} \in \mathbb{R}^H_+$ at $t = 0$, and $e_{is} \in \mathbb{R}^H_+$, in each expected state, $s \in S_i$, if it prevails. Given prices and expectations, $\varpi := ((p_0, q), p := (p_s)) \in \mathbb{R}^H_+ \times \mathbb{R}^J \times P$, the generic i^{th} agent's consumption set is $(\mathbb{R}^H_+)^{S'_i}$ and her budget set is:

$$B_i(\varpi, V) := \left\{ (x := (x_s), z) \in (\mathbb{R}^H_+)^{S'_i} \times \mathbb{R}^J : \left[\begin{array}{c} p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot z \\ p_s \cdot (x_s - e_{is}) \leqslant V(p,s) \cdot z,\ \forall s \in S_i \end{array} \right. \right\}.$$

Each consumer, $i \in I$, has preferences, $\prec_i$, represented, for each $x \in (\mathbb{R}^H_+)^{S'_i}$, by the set, $P_i(x) := \{y \in (\mathbb{R}^H_+)^{S'_i} : x \prec_i y\}$, of consumptions which are strictly preferred to x. In the above economy, denoted $\mathcal{E} = \{(I, S, H, J), V, (S_i)_{i \in I}, (e_i)_{i \in I}, (\prec_i)_{i \in I}\}$, agents optimise their consumptions in the budget sets. So the concept of equilibrium:

Definition 1 *A collection of prices,* $\varpi := ((p_0, q), p := (p_s)) \in \mathbb{R}^H_+ \times \mathbb{R}^J \times P$, *and strategies,* $[(x_i, z_i)] \in \times_{i \in I} B_i(\varpi, V)$, *is an equilibrium of the economy,* $\mathcal{E}$, *if the following holds:*

(a) $\forall i \in I,\ (x_i, z_i) \in B_i(\varpi, V)$ *and* $P_i(x_i) \times \mathbb{R}^J \cap B_i(\varpi, V) = \varnothing$*;*

(b) $\sum_{i\in I}(x_{is}-e_{is})=0,\ \forall s\in \underline{\mathbf{S}}'$;

(c) $\sum_{i\in I} z_i=0$.

The economy, $\mathcal{E}$, is called standard under the following conditions:

Assumption A1 *(monotonicity):* $\forall (i,x,y)\in I\times(\mathbb{R}_+^{HS_i'})^2,\ (x\leqslant y,\ x\neq y)\Rightarrow (x\prec_i y)$;

Assumption A2 *(strong survival):* $\forall i\in I,\ e_i\in \mathbb{R}_{++}^{HS_i'}$;

Assumption A3: $\forall i\in I$, $\prec_i$ *is lower semicontinuous convex-open-valued and such that* $x\prec_i x+\lambda(y-x)$, *whenever* $(x,y,\lambda)\in \mathbb{R}_+^{HS_i'}\times P_i(x)\times]0,1]$.

3 The existence Theorem and proof

Theorem 1 *Generically in the set of assets' payoffs, in cash value at market prices, and in realizable states only, a standard economy,* $\mathcal{E}$, *admits an equilibrium.*

From Claim 1-(vii) and its proof, Theorem 1 will be demonstrated if we show that equilibrium exists if the economy's financial structure is represented by an arbitrary element $\widetilde{V}\in\Lambda_\lambda:=\{M\in\mathcal{V}_\lambda : M\in\mathcal{V}_\lambda^p,\ \forall p\in P\}\neq\varnothing$, for some $\lambda>0$. Hereafter, we set as given such elements, $\lambda>0$ and $\widetilde{V}\in\Lambda_\lambda$, and show they yield an equilibrium.

3.1 Bounding the economy

For every $(i,\varpi:=((p_0,q),p))\in I\times P_0\times P$, we let:

$$\overline{B}_i(\varpi,\widetilde{V}):=\left\{\ (x,z)\in(\mathbb{R}_+^H)^{S_i'}\times\mathbb{R}^J:\left[\begin{array}{c} p_0\cdot(x_0-e_{i0})\leqslant -q\cdot z+1 \\ p_s\cdot(x_s-e_{is})\leqslant \widetilde{V}(p,s)\cdot z+1,\ \forall s\in S_i\end{array}\right.\right\};$$

$$\overline{\mathcal{A}}(\varpi,\widetilde{V}):=\{[(x_i,z_i)]\in\times_{i\in I}\overline{B}_i(\varpi,\widetilde{V}):\ \textstyle\sum_{i\in I}(x_{is}\text{-}e_{is})=0,\ \forall s\in\underline{\mathbf{S}}',\ \sum_{i\in I}z_i=0\}.$$

Lemma 1 $\exists r>0:\forall\varpi\in P_0\times P,\ \forall[(x_i,z_i)]\in\overline{\mathcal{A}}(\varpi,\widetilde{V}),\ \|[(x_i,z_i)]\|<r$

Proof : see the Appendix. □

Lemma 1 permits to bound the economy. Thus, we define (along Lemma 1), for every $\varpi := ((p_0, q), p) \in P_0 \times P$, the following convex compact sets:

$X_i := \{x \in (\mathbb{R}^H_+)^{S'_i} : \|x\| \leqslant r\}$, $Z := \{z \in \mathbb{R}^J : \|z\| \leqslant r\}$ and $\mathcal{A}(\varpi) := \overline{\mathcal{A}}(\varpi, \widetilde{V}) \cap (\times_{i \in I} X_i \times Z)$.

3.2 The existence proof

For every $i \in I$ and every $\varpi := ((p_0, q), p) \in P_0 \times P$, we let:

$$B'_i(\varpi) := \{(x, z) \in X_i \times Z : p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot z + \gamma_{(p_0,q)} \text{ and } p_s \cdot (x_s - e_{is}) \leqslant \widetilde{V}(p, s) \cdot z + \gamma_{(s,p_s)}, \forall s \in S_i\};$$

$$B''_i(\varpi) := \{(x, z) \in X_i \times Z : p_0 \cdot (x_0 - e_{i0}) < -q \cdot z + \gamma_{(p_0,q)} \text{ and } p_s \cdot (x_s - e_{is}) < \widetilde{V}(p, s) \cdot z + \gamma_{(s,p_s)}, \forall s \in S_i\},$$

where $\gamma_{(p_0,q)} := 1 - \|(p_0, q)\|$, $\gamma_{(s,p_s)} := 1 - \|p_s\|$, $\forall s \in \underline{\mathbf{S}}$ and $\gamma_{(s,p_s)} := 0$, $\forall s \in S \backslash \underline{\mathbf{S}}$.

Claim 2 *For every* $(i, \varpi := ((p_0, q), p)) \in I \times P_0 \times P$, $B''_i(\varpi) \neq \varnothing$.

Proof Let $i \in I$ and $\varpi := ((p_0, q), p) \in P_0 \times P$ be given. From Assumption *A2*, we may choose $x \in X_i$, which meets all budget constraints, and with a strict inequality in each state $s \in S_i$, such that $p_s \neq 0$. If $p_0 \neq 0$, or $(p_0, q) = 0$, then, $(x, 0) \in B''_i(\varpi)$. Finally, if $p_0 = 0$ and $q \neq 0$, then, for $N \in \mathbb{N}$ big enough, $(x, -q/N) \in B''_i(\varpi)$. □

Claim 3 *For all* $(i, (p_0, q), p) \in I \times P_0 \times P$, B''_i *is lower semicontinuous.*

Proof Let $(i, \varpi := ((p_0, q), p)) \in I \times P_0 \times P$ be given. The convexity of $B''_i(\varpi)$ is obvious and implies, from Claim 2, $B'_i(\varpi) = \overline{B''_i(\varpi)}$. From the relation $\widetilde{V} \in \mathcal{V}^p_\lambda$ and Claim 1, B''_i is lower semicontinuous, as standard, for having a local open graph. □

Claim 4 *For every* $(i, (p_0, q), p)) \in I \times P_0 \times P$, B'_i *is upper semicontinuous.*

Proof Let $(i, \varpi := ((p_0, q), p)) \in I \times P_0 \times P$ be given. B'_i is (as standard) upper semicontinuous at ϖ, for having a closed graph in a compact set. □

We introduce additional fictious agents for markets and a reaction correspondence for each agent, defined on the convex compact set, $\Theta := P_0 \times P \times (\times_{i \in I} X_i \times Z)$. Thus, we let, for each $i \in I$ and every $\theta := (\varpi := ((p_0, q), p),\ (x, z) := [(x_i, z_i)]) \in \Theta$:

$$\Psi_0(\theta) := \{[(p'_0, q'), p] \in P_0 \times P : \sum_{s\in\underline{\mathbf{S}}'}(p'_s\text{-}p_s)\cdot \sum_{i\in I}(x_{is}\text{-}e_{is}) + (q'\text{-}q))\cdot \sum_{i\in I} z_i > 0\};$$

$$\Psi_i(\theta) := \left\{ \begin{array}{lll} B'_i(\varpi) & if & (x_i, z_i) \notin B'_i(\varpi) \\ B''_i(\varpi) \cap P_i(x_i) \times Z & if & (x_i, z_i) \in B'_i(\varpi) \end{array} \right\};$$

Claim 5 *For each $i \in I \cup \{0\}$, Ψ_i is lower semicontinuous.*

Proof The correspondences Ψ_0 is lower semicontinuous for having an open graph. We recall from Claim 1 that $\Psi_i(\theta)$ will not yield any fall in rank problem in budget sets, since in all cases, $\widetilde{V} \in \mathcal{V}^p_\lambda$, and that agents' anticipations will never vary along De Boisdeffre (2016), also from Claim 1, since markets always preclude arbitrage.

- Assume that $(x_i, z_i) \notin B'_i(\varpi)$. Then, $\Psi_i(\theta) = B'_i(\varpi)$.

Let V be an open set in $X_i \times Z$, such that $V \cap B'_i(\varpi) \neq \varnothing$. It follows from the convexity of $B'_i(\varpi)$ and the non-emptyness of the open set $B''_i(\varpi)$ that $V \cap B''_i(\varpi) \neq \varnothing$. From Claims 1 and 3, there exists a neighborhood U of ϖ, such that $V \cap B'_i(\varpi') \supset V \cap B''_i(\varpi') \neq \varnothing$, for every $\varpi' \in U$.

Since $B'_i(\varpi)$ is nonempty, closed, convex in the compact set $X_i \times Z$, there exist two open sets V_1 and V_2 in $X_i \times Z$, such that $(x_i, z_i) \in V_1$, $B'_i(\varpi) \subset V_2$ and $V_1 \cap V_2 = \varnothing$. From Claims 1 and 4, there exists a neighborhood $U_1 \subset U$ of (ϖ), such that $B'_i(\varpi') \subset V_2$, for every $\varpi' \in U_1$. Let $W = U_1 \times (\times_{j\in I} W_j)$, where $W_i := V_1$ and $W_j := X_j \times Z$, for every $j \in I\backslash\{i\}$. Then, W is a neighborhood of θ, such that $\Psi_i(\theta') = B'_i(\varpi')$, and, from above, $V \cap \Psi_i(\theta') \neq \varnothing$, for every $\theta' := (\varpi', (x', z')) \in W$. Thus, Ψ_i is lower semicontinuous at θ.

- Assume that $(x_i, z_i) \in B'_i(\varpi)$, i.e., $\Psi_i(\theta) = B''_i(\varpi) \cap P_i(x) \times Z$.

Lower semicontinuity is immediate if $\Psi_i(\theta) = \varnothing$. Assume $\Psi_i(\theta) \neq \varnothing$. We recall that P_i (from Assumption *A3*) is lower semicontinuous with open values and that B''_i has a local open graph. As corollary & from Claim 1, the correspondence $(\varpi',(x',z')) \in \Theta \to B''_i(\varpi') \cap P_i(x'_i) \times Z \subset B'_i(\varpi')$ is lower semicontinuous at θ. Then, from Claim 1 and the latter inclusions, Ψ_i is lower semicontinuous at θ. □

Claim 6 *There exists* $\theta^* := (\varpi^* := ((p_0^*, q^*), p^*),\ [(x_i^*, z_i^*)]) \in \Theta$*, such that*:

(i) $\forall((p_0,q),p) \in P_0 \times P,\ \sum_{s \in \underline{\mathbf{S}}'} (p_s^* - p_s) \cdot \sum_{i \in I} (x_{is}^* - e_{is}) + (q^* \text{-} q) \cdot \sum_{i \in I} z_i^* \geqslant 0$;

(ii) $\forall i \in I,\ (x_i^*, z_i^*) \in B'_i(\varpi^*)$ *and* $B''_i(\varpi^*) \cap P_i(x_i^*) \times Z = \varnothing$.

Proof Quoting Gale-Mas-Colell, 1975-79 [9,10]: "Given $X = \times_{i=1}^m Xi$, where X_i is a non-empty compact convex subset of $\mathbb{R}^n$, let $\varphi_i : X \to X_i$ be m convex (possibly empty) valued correspondences, which are lower semicontinuous. Then there exists x in X such that for each i either $x_i \in \varphi_i(x)$ or $\varphi_i(x) = \varnothing$".

The correspondences, $\Psi_0 : \Theta \to P_0 \times P$, $\Psi_i : \Theta \to X_i \times Z$ (for each $i \in I$) satisfy the conditions of the above theorem and yield Claim 6. □

Claim 7 $\sum_{i=1}^m z_i^* = 0$.

Proof Assume, by contraposition, that $\sum_{i=1}^m z_i^* \neq 0$. Then, from Claim 6-$(i)$, $(p_0^* - p_0) \cdot \sum_{i \in I} (x_{i0}^* \text{-} e_{i0}) + q \cdot \sum_{i=1}^m z_i^* \leqslant q^* \cdot \sum_{i=1}^m z_i^*$, for every $(p_0, q) \in P_0$, which implies $q^* \cdot \sum_{i=1}^m z_i^* > 0$ and $\gamma_{(p_0,q)} = 0$. From Claim 6-(ii), the relations $(x_i^*, z_i^*) \in B'_i(\varpi^*)$ hold, for each $i \in I$, whose budget constraint in state $s = 0$ is $p_0^* \cdot (x_{i0}^* - e_{i0}) \leqslant -q^* \cdot z_i^*$. Adding them up yields $p_0^* \cdot \sum_{i \in I} (x_{i0}^* - e_{i0}) \leqslant -q^* \cdot \sum_{i=1}^m z_i^* < 0$, which contradicts Claim 6-(i). □

Claim 8 $\sum_{i \in I} (x_{is}^* - e_{is}) = 0, \forall s \in \underline{\mathbf{S}}' := \cap_i S'_i$.

Proof Let $s \in \underline{\mathbf{S}}'$ be given, say $s \in \underline{\mathbf{S}}$, and assume that $\sum_{i \in I} (x_{is}^* - e_{is}) \neq 0$. Applying Claim 6 to good prices yields: $p_s^* \cdot \sum_{i \in I} (x_{is}^* \text{-} e_{is}) > 0$ and $\gamma_{(s,p_s)} = 0$. Then, added budget constraints in Claims 6-7 yield: $0 < p_s^* \cdot \sum_{i \in I} (x_{is}^* \text{-} e_{is}) \leqslant \widetilde{V}(s, p_s^*) \cdot \sum_{i \in I} z_i^* = 0$. □

Claim 9 $(x^*, z^*) := [(x_i^*, z_i^*)] \in \mathcal{A}(\varpi^*)$*, hence,* $\|(x^*, z^*)\| < r$.

Proof Claim 9 follows immediately from Claims 6-7-8 and Lemma 1 above. $\square$

Claim 10 *For each* $i \in I$*,* (x_i^*, z_i^*) *is optimal in* $B_i'(\varpi^*)$.

Proof Let $i \in I$ be given. Assume, by contraposition, there exists $(x_i, z_i) \in B_i'(\varpi^*) \cap P_i(x_i^*) \times Z$. From Claim 9, the relations $\|x_i^*\| < r$ and $\|z_i^*\| < r$ hold and, from Assumption *A3*, the relations $\|x_i\| < r$ and $\|z_i\| < r$ may be assumed.

From Claim 3, there exists $(x_i', z_i') \in B_i''(\varpi^*) \subset B_i'(\varpi^*)$. By construction, $(x_i^n, z_i^n) := [\frac{1}{n}(x_i', z_i') + (1 - \frac{1}{n})(x_i, z_i)] \in B_i''(\varpi^*)$, for every $n \in \mathbb{N}$. From Assumption *A3*, $(x_i^N, z_i^N) \in P_i(x_i^*) \times Z$ holds, for $N \in \mathbb{N}$ big enough. Hence, $(x_i^N, z_i^N) \in B_i''(\varpi^*) \cap P_i(x_i^*) \times Z$, which contradicts Claim 6-(ii). $\square$

Claim 11 $\gamma_{(p_0,q)} = 0$*, i.e.,* $\|(p_0^*, q^*)\| = 1$*, and* $\gamma_{(s,p_s)} = 0$*, i.e.,* $\|p_s^*\| = 1$*,* $\forall s \in \underline{\mathbf{S}}$*.*
Hence, $B_i'(\varpi^*) = B_i(\varpi^*, \widetilde{V})$*, for every* $i \in I$*, where:*

$$B_i(\varpi^*, \widetilde{V}) := \{\ (x := (x_s), z) \in (\mathbb{R}_+^H)^{S_i'} \times \mathbb{R}^J : \left\langle \begin{array}{l} p_0^* \cdot (x_0 - e_{i0}) \leqslant -q^* \cdot z \\ p_s^* \cdot (x_s - e_{is}) \leqslant \widetilde{V}(p^*, s) \cdot z,\ \forall s \in S_i \end{array} \right. \}.$$

Proof Let $(i, s) \in I \times \underline{\mathbf{S}}'$ be given, say $s = 0$, the proof being the same for $s \in \underline{\mathbf{S}}$.

From Claim 6, the relation $p_0^* \cdot (x_{i0}^* - e_{i0}) \leqslant q^* \cdot z_i^* + \gamma_{(p_0,q)}$ holds. Assume, by contraposition, that $p_0^* \cdot (x_{i0}^* - e_{i0}) < -q^* \cdot z_i^* + \gamma_{(p_0,q)}$. From Claim 9, $\|x_i^*\| < r$, and, from Assumptions *A1-A3*, there exists $x_i \in P_i(x_i^*)$ (differing from x_i^* in x_{i0} only), close to x_i^* so that $p_0^* \cdot (x_{i0}\text{-}e_{i0}) \leqslant -q^* \cdot z_i^* + \gamma_{(p_0,q)}$. This contradiction to Claim 10 insures that $p_0^* \cdot (x_{i0}^* - e_{i0}) = q^* \cdot z_i^* + \gamma_{(p_0,q)}$ holds for each $i \in I$. Then, Claim 9 yields the relations: $0 = p_s^* \cdot \sum_{i \in I} (x_{is}^* \text{-} e_{is}) = -q \cdot \sum_{i \in I} z_i^* + \#I\ \gamma_{(p_0,q)} = \#I\ \gamma_{(p_0,q)}$. $\square$

Claim 12 $(p_0^*, q^*, p^*, \widetilde{V}, [(x_i^*, z_i^*)])$ *is an equilibrium, hence, Theorem 1 holds.*

Proof The collection $(p_0^*, q^*, p^*, \widetilde{V}, [(x_i^*, z_i^*)])$ meets all Conditions of Definition 1 of equilibrium of an economy, whose financial structure is $\widetilde{V}$. Theorem 1 is proved. $\square$

Remark Theorem 1 may surprise. It reduces set of parameters to assets' cash payoffs (instead of payoffs and endowments); it applies to asymmetric information and non-odered preferences; it yields normalized (instead of unknown) spot prices at equilibrium; it uses simple nominal asset techniques of proof. The reason for this is the good behaviour of payoff matrixes under Claim 1. In fact, the non-semicontinuity of demand correspondences, that may result from a fall in rank problem a la Hart (1975), is not binding. It can always be circumvented, owing to Claim 1-(vii).

4 The existence Theorem with numeraire assets

We consider an economy, where assets only pay in the same (bundle of) commodities, $e \in \mathbb{R}_+^H$ (we let $\|e\| = 1$), in any state. These assets are referred to as numeraire. The economy is in anything alike that of Section 2, but the fact that there exists a given $S \times J$ matrix, V, of payoffs in numeraire, e, across states. This matrix, V, can also be written as an element of the set of $(S \times H') \times J$ payoff matrixes. Agents and markets have the same charateristics as above, and we resume all notations and assumptions of Section 2. Moreover, agents' preferences are now represented by continuous, strictly concave, strictly increasing functions, $u_i : X_i \to \mathbb{R}$, for each $i \in I$.

From Claim 12 and Theorem 1, above, for every $n \in \mathbb{N}$, there exists an equilibrium, $\mathcal{C}^n := (p_0^n, q^n, p^n := (p_s^n), \widetilde{V}^n, (x^n, z^n) := [(x_i^n, z_i^n)])$, for some payoff matrix $\widetilde{V}^n \in \mathcal{V}_{1/n}$ along Claim 1. From compactness, we may assume the price sequence, $\{(p_0^n, q^n, p^n := (p_s^n))\}$, converges to some $(p_0^*, q^*, p^* := (p_s^*)) \in P_0 \times P$, such that $\|(p_0^*, q^*)\| = 1$ and $\|p_s^*\| = 1$, for each $s \in \underline{\mathbf{S}}$, while $\{\widetilde{V}^n\}$ converges to V.

Without an additional assumption, nothing prevents the fall to zero of the value of the numeraire $(p_s^* \cdot e)$, in some state $s \in \underline{\mathbf{S}}$, and a subsequent arbitrage problem. This is resolved by assuming that agents' utilities are separable, that is, for each $i \in I$, there exist continuous utility indexes, $u_i^s : \mathbb{R}_+^2 \to \mathbb{R}$ (for $s \in S_i$), such that $u_i(x) = \sum_{s \in S_i} u_i^s(x_0, x_s)$, for every $x \in (\mathbb{R}_+^H)^{S'_i}$. Moreover, we assume, non restrictively, that agents' signals, (S_i), embed the information markets reveal, along De Boisdeffre (2016), therefore that the payoff and information structure, $[V, (S_i)]$, is arbitrage-free, along the latter paper's definition. To shorten exposition, but w.l.o.g., we finally assume that the $\underline{\mathbf{S}} \times J$ payoff matrix, $V_{\underline{\mathbf{S}}}$, has full column rank.[2]

The above equilibrium sequence, $\{\mathcal{C}^n\}$, satisfies the following properties.

Lemma 2 *The following Assertions hold*:

(i) $\forall (n, i, s) \in \mathbb{N} \times I \times \underline{\mathbf{S}}'$, $x_{is}^n \in [0, E]^H$, *where* $E := \max_{(s,h) \in S' \times H} \sum_{i \in I} e_{is}^h$;

(ii) it may be assumed to exist $(x^*, z^*) := [(x_i^*, z_i^*)] = \lim_{n \to \infty} [(x_i^n, z_i^n)]$;

(iii) for each $s \in \underline{\mathbf{S}}'$, $\sum_{i \in I} (x_{is}^* - e_{is}) = 0$ *and, moreover,* $\sum_{i \in I} z_i^* = 0$.

Proof Assertion (i) is standard, from market clearance conditions of equilibrium. Assertion (ii): the fact that the sequence $\{(x^n, z^n)\}$ is bounded, hence may be assumed to converge, results from Lemma 1 (see the Appendix). And Assertion (iii) results from the market clearance conditions on $\{\mathcal{C}^n\}$, passing to the limit. □

And we show the following full existence Theorem.

Theorem 2 *The above collection,* $\mathcal{C}^* := (p_0^*, q^*, p^*, (x^*, z^*))$, *of prices, expectations & strategies is an equilibrium of the numeraire asset economy with payoff matrix* V.

Proof Let us define $\mathcal{C}^* := (p_0^*, q^*, p^*, (x^*, z^*))$ as above. From Lemma 2-(ii)-(iii), $\mathcal{C}^*$ meets Conditions (b)-(c) of Definition 1 of equilibrium. Thus, it suffices to show that

[2] The latter assumption can easily be dropped but this lengthens the proof.

the relations, $[(x_i^*, z_i^*)] \in \times_{i\in I} B_i(\varpi^*)$, of Section 2 (where $\varpi := ((p_0^*, q^*), p^*) \in P_0{\times}P$), and Definition 1-$(a)$ hold.

Let $i \in I$ be given. From the definition, the relations $p_0^n{\cdot}(x_{i0}^n - e_{i0}) \leqslant -q^n{\cdot}z_i^n$ hold, for all $n \in \mathbb{N}$, and, yield $p_0^*{\cdot}(x_{i0}^*\text{-}e_{i0}) \leqslant -q^*{\cdot}z_i^*$, in the limit. Similarly, and from standard continuity arguments, the relations $x_i^* \in X_i$ and $p_s^*{\cdot}(x_{is}^* - e_{is}) \leqslant V(p^*, s){\cdot}z_i^*$ also hold for each $s \in S_i$. Hence, $[(x_i^*, z_i^*)] \in \times_{i\in I} B_i(\varpi^*)$.

We now assume, by contraposition, that $\mathcal{C}^*$ fails to meet Condition (a) of Definition 1, that is, there exist $i \in I$, $(x, z) \in B_i(\varpi^*)$ and $\varepsilon \in \mathbb{R}_{++}$, such that:

(I) $\qquad \varepsilon + u_i(x_i^*) < u_i(x).$

We may assume: $\qquad (II) \qquad \exists\ (\delta, M) \in \mathbb{R}^2_{++}$: $x_s \in [\delta, M]^H,\ \forall s \in S_i.$

Indeed, the upper bound, M, exists from the definition of x and, for $\alpha \in]0,1]$ small enough, the strategy $(x^\alpha, z^\alpha) := ((1-\alpha)x + \alpha e_i, (1-\alpha)z) \in B_i(\varpi^*)$ meets both relations (I) and (II), along Assumption *A1* and the continuity of the mapping $\alpha \in [0,1] \mapsto u_i(x^\alpha)$. So, relation (II) may be assumed. Then, it is immediate from the relations (I)-(II) and $(x, z) \in B_i(\varpi^*)$, from Lemma 2, Assumptions *A2*-*A3* and continuity arguments, that we may also assume there exists $\gamma \in \mathbb{R}_{++}$, such that:

(III) $\quad p_0^*{\cdot}(x_0 - e_{i0}) \leqslant -q^*{\cdot}z$ and $p_s^*{\cdot}(x_s - e_{is}) \leqslant -\gamma + V(p^*, s){\cdot}z$, $\forall s \in S_i$.

From relations (I)-(II)-(III), we may also assume there exists $\gamma' \in]0, \gamma[$, such that:

(IV) $\quad p_0^*{\cdot}(x_0 - e_{i0}) \leqslant -\gamma' - q^*{\cdot}z$ and $p_s^*{\cdot}(x_s - e_{is}) \leqslant -\gamma' + V(p^*, s){\cdot}z$, $\forall s \in S_i$.

Indeed, the above assertion is obvious, from relations (III), if $p_0^*{\cdot}(x_0 - e_{i0}) < -q^*{\cdot}z$. Assume that $p_0^*{\cdot}(x_0 - e_{i0}) = -q^*{\cdot}z$. If $p_0^* = 0$, then, $q^* \neq 0$, and relations (IV) hold if we replace z by $z - q^*/N$, for $N \in \mathbb{N}$ big enough. If $p_0^* \neq 0$ and $x_0 \neq 0$, the desired assertion results from Assumption *A1* and above. Otherwise, $-q^* \cdot z = -p_0^* \cdot e_{i0} < 0$, and a slight change in portfolio insures relations (IV). From relations (IV), the continuity of the scalar product and Lemma 2, there exists $N_1 \in \mathbb{N}$, such that, for every $n \geqslant N_1$:

$$(V) \qquad p_0^n \cdot (x_0 - e_{i0}) < -q^n \cdot z \quad \text{and} \quad p_s^n \cdot (x_s - e_{is}) < \widetilde{V}^n(p^n, s) \cdot z, \ \forall s \in S_i.$$

Along relations (V), Assumption *A1-A3*, Lemma 2 and the definition of equilibrium, there exists $N_2 > N_1$, such that: $(VI) \qquad u_i(x) \leqslant u_i(x_i^n) < \varepsilon + u_i(x_i^*)$, $\forall n \geqslant N_2$.

Let $n \geqslant N_2$ be given. Then, Conditions (I)-(VI) yield: $u_i(x) < \varepsilon + u_i(x_i^*) < u_i(x)$.

This contradiction proves that $\mathcal{C}^*$ is an equilibrium and Theorem 2 holds. □

Appendix

Lemma 1 $\exists r > 0 : \forall \varpi \in P_0 \times P, \ \forall [(x_i, z_i)] \in \overline{\mathcal{A}}(\varpi, \widetilde{V}), \ \|[(x_i, z_i)]\| < r$

We start with Lemma 1 in the general setting of of Section 3.

Proof Let $\varpi := ((p_0, q), p) \in P_0 \times P$, and $[(x_i, z_i)] \in \overline{\mathcal{A}}(\varpi) := \overline{\mathcal{A}}(\varpi, \widetilde{V})$ be given.

- As seen under Lemma 2, the relations, $x_{is} \in [0, E]^H$, where $E := \max_{(s,h) \in S' \times H} \sum_{i \in I} e_{is}^h$, hold, for every $(i, s) \in I \times \underline{\mathbf{S}}'$, from market clearance conditions.
- From above and the relation $(\overline{p}_s) \in (\mathbb{R}_{++}^H)^{S \setminus \underline{\mathbf{S}}'}$, Lemma 1 will be proved if the portfolios, (z_i), are bounded independently of ϖ. We now prove that property.
- Let $\delta = 1 + (\|(\overline{p})\| + 1)\|(e_i)\|$. Assume, by contraposition, that, for every $n \in \mathbb{N}$, there exists $[(x_i^n, z_i^n)] \in \overline{\mathcal{A}}(\varpi^n)$, for some $\varpi^n := ((p_0^n, q^n), p^n) \in P_0 \times P$, such that $\|z^n\| := \|(z_i^n)\| > n$. For each $n \in \mathbb{N}$, market clearance in $[(x_i^n, z_i^n)] \in \overline{\mathcal{A}}(\varpi^n)$ yields:

$\sum_{i\in I}\ z_i^n = 0$, and $\widetilde{V}(p^n, s_i)\cdot z_i^n \geqslant -\delta,\ \forall(i, s_i, n) \in I\times S_i\times\mathbb{N}$.

- We may assume $\lim_{n\to\infty}\ \varpi^n = \varpi^* := ((p_0^*, q^*), p^*) \in P_0\times P$.

- For every $(i,n) \in I\times\mathbb{N}$, we let $x_i'^n := \frac{x_i^n}{\|z^n\|} + (1 - \frac{1}{\|z^n\|})e_i$ and $z_i'^n := \frac{z_i^n}{\|z^n\|}$. Then, the relations $[(x_i'^n, z_i'^n)] \in \overline{\mathcal{A}}(\varpi^n)$ and $\|(z_i'^n)\| = 1$ hold and the sequence $\{[(x_i'^n, z_i'^n)]\}_{n\in\mathbb{N}}$ has a cluster point, $[(x_i, z_i)]$, such that $\|(z_i)\| = 1$, and satisfies the relations:

  $\sum_{i\in I}\ z_i'^n = 0$, $\widetilde{V}(p^n, s_i)\cdot z_i'^n \geqslant -\delta/n, \forall(i, s_i, n) \in I\times S_i\times\mathbb{N}$, and, passing to the limit, $\sum_{i\in I}\ z_i = 0$, $\widetilde{V}(p^*, s_i)\cdot z_i \geqslant 0, \forall(i, s_i) \in I\times S_i$,

Since $\widetilde{V} \in \Lambda_\lambda$, the latter relations imply $(z_i) = 0$, from Claim 1-(v), and contradict the fact that $\|(z_i)\| = 1$. This contradiction ends the proof. □

We proceed with Lemma 1 for the numeraire asset economy.

Proof

- As above, we need only show portfolios are bounded, but, then, accross all economies, $\mathcal{E}^n = \{(I, S, H, J), \widetilde{V}^n, (S_i)_{i\in I}, (e_i)_{i\in I}, (u_i)_{i\in I}\}$. We let the reader check that all contraposition arguments above translate, mutatis mutandis, to double indexed sequences of prices, $\varpi^{(n,k)} := ((p_0^{(n,k)}, q^{(n,k)}), p^{(n,k)}) \in P_0\times P$, and strategies $[(x_i^{(n,k)}, z_i^{(n,k)})] \in \overline{\mathcal{A}}(\varpi^{(n,k)}, \widetilde{V}^n)$, where $(n,k) \in \mathbb{N}^2$ $\big(n$ standing for the economy$\big)$, whose final contraposition arguments are:

$\sum_{i\in I}\ z_i'^{(n,k)} = 0$, $\widetilde{V}^n(p^n, s_i)\cdot z_i'^{(n,k)} \geqslant -\delta/k, \forall(i, s_i, n, k) \in I\times S_i\times\mathbb{N}^2$, and, in the limit, $\sum_{i\in I}\ z_i = 0$, $V(p^*, s_i)\cdot z_i \geqslant 0, \forall(i, s_i) \in I\times S_i$, with $\|(z_i)\| = 1$.

- Along the model's specification, the latter relations will imply $(z_i) = 0$, from Claim 1-(v), hence, the same contradiction as above, whenever $p_s^* \cdot e > 0$ holds for every $s \in \underline{\mathbf{S}}$. Lemmata 1, below, shows this latter property indeed holds.

First, we introduce new notations and let, for all $(i,s,x) \in I \times \underline{\mathbf{S}} \times (\mathbb{R}^H_+)^{S'_i}$:

- $y \succ^i_s x$ denote a consumption, such that $u_i(y) > u_i(x)$ and $y_{s'} = x_{s'}$, $\forall s' \in S'_i \backslash \{s\}$;
- $\mathcal{A} := \{(x_i) \in \times_{i\in I}(\mathbb{R}^H_+)^{S'_i} : \sum_{i\in I}\ x_{is} = \sum_{i\in I}\ e_{is},\ \forall s \in \underline{\mathbf{S}}'\}$;
- $P_s := \{p \in \mathbb{R}^H_+,\ \|p\| = 1 : \exists i \in I,\ \exists (x_i) \in \mathcal{A},\ \textit{such that}\ (y \succ^i_s x_i) \Rightarrow (p{\cdot}y_s \geqslant p{\cdot}x_{is} \geqslant p{\cdot}e_{is})\}$.

Lemmata 1 *The following Assertions hold:*

(i) $\forall s \in \underline{\mathbf{S}},\ P_s$ *is a compact set;*

(ii) $\exists\ \delta > 0 : \forall (s,p) \in \underline{\mathbf{S}} \times P_s,\ p \cdot e \geqslant \delta$*;*

(iii) $\forall (n,s) \in \mathbb{N} \times \underline{\mathbf{S}},\ p^n_s \in P_s$*, hence,* $p^*_s \cdot e \geqslant \delta > 0$.

Proof Assertion (i) Let $s \in \underline{\mathbf{S}}$ and a converging sequence $\{p^k\}_{k\in\mathbb{N}}$ of elements of P_s be given. Its limit, $p = \lim_{k\to\infty}\ p^k$, satisfies $\|p\| = 1$. We may assume there exist (a same) $i \in I$ and a sequence, $\{x^k\}_{k\in\mathbb{N}} := \{(x^k_i)\}_{k\in\mathbb{N}}$, of elements of $\mathcal{A}$, converging to some $x := (x_i)$ in the closure of $\mathcal{A}$ in $\times^m_{i=1}(\mathbb{R}_+ \cup \{+\infty\})^{LS'_i}$, such that, for each $k \in \mathbb{N}$, (p^k_s, i, x^k) satisfies the conditions of the definition of P_s. From Lemma 2-(i) $\{(x^k_{is'})\}_{k\in\mathbb{N}}$, is bounded, hence, $x_{s'} := (x_{is'})$ is finite, for each $s' \in \underline{\mathbf{S}}'$.

For every $k \in \mathbb{N}$, let $\widetilde{x}^k := (\widetilde{x}^k_i) \in \mathcal{A}$ be defined by $(\widetilde{x}^k_{i0}) := (x_{i0})$, $(\widetilde{x}^k_{is}) := (x_{is})$ and $(\widetilde{x}^k_{is'}) := (x^k_{is'})$, for each $s' \in S_i \backslash \{s\}$. Then, the relations $p^k \cdot (x^k_{is} - e_{is}) \geqslant 0$, for every $k \in \mathbb{N}$, yield, in the limit, $p{\cdot}(\widetilde{x}^k_{is} - e_{is}) := p{\cdot}(x_{is} - e_{is}) \geqslant 0$. We now show there exists $k \in \mathbb{N}$, such that $(p, i, \widetilde{x}^k)$ satisfies the conditions of the definition of P_s (hence, $p := \lim p^k \in P_s$). By contraposition, assume the contrary, i.e., for each $k \in \mathbb{N}$, there exists $y^k \in (\mathbb{R}^L_+)^{S'_i}$, such that $y^k_{s'} = \widetilde{x}^k_{is'}$, for each $s' \in S'_i \backslash \{s\}$, $u_i(y^k) > u_i(\widetilde{x}^k_i)$ and $p{\cdot}(y^k_s - x_{is}) < 0$. Then, given $k \in \mathbb{N}$, there exists (from Assumption *A3* and separability) $K \geqslant k$, such that, for every $k' \geqslant K$, $u_i(y^k) > u_i(x^{k'}_i)$. The latter relations imply, by construction of each $x^{k'}$ (for $k' \geqslant K$), $p^{k'}_s \cdot (y^k_s\text{-}x^{k'}_{is}) \geqslant 0$, hence, in the limit $(k' \to \infty)$, $p{\cdot}(y^k_s - x_{is}) \geqslant 0$, contradicting the inequality, $p{\cdot}(y_s^k - x_{is}) < 0$, assumed above. This contradiction proves that $p \in P_s$, hence, P_s is closed, therefore, compact. $\square$

Assertion (ii) Let $s \in \underline{\mathbf{S}}$ and $p \in P_s$ be given. We prove, first, that $p \cdot e > 0$. Indeed, let $(p, i, x) \in P_s \times I \times \mathcal{A}$ meet the conditions of the definition of P_s. From Assumption *A2*, there exists $a_i \in (\mathbb{R}_+^L)^{S_i'}$ such that, $a_{is'} = 0$, for each $s' \in S_i' \backslash \{s\}$, and $p \cdot a_{is} < p \cdot e_{is} \leqslant p \cdot x_{is}$. Then, for every $n > 1$, we let $x_i^n := (\frac{1}{n} a_i + (1 - \frac{1}{n}) x_i) \in (\mathbb{R}_+^L)^{S_i'}$, which satisfies $p_s \cdot (x_{is}^n - x_{is}) < 0$ by construction. Let $E_i^s \in (\mathbb{R}_+^L)^{S_i'}$ be defined by $E_{is}^s = e$, $E_{is'}^s = 0$, for $s' \neq s$. Along Assumptions *A1-A3*, there exists $n > 1$, such that $y := (x_i^n + (1 - \frac{1}{n}) E_i^s)$ satisfies $u(y) > u(x_i)$, which implies, $p \cdot x_{is} \leqslant p \cdot y_s = p \cdot (x_{is}^n + (1 - \frac{1}{n}) e) < p \cdot x_{is} + (1 - \frac{1}{n}) p \cdot e$. Hence, $p \cdot e > 0$. The mapping $\varphi_s : P_s \to \mathbb{R}_{++}$, defined by $\varphi_s(p) := p{\cdot}e$ is continuous and attains its minimum for some element $\underline{p}$ on the compact set P_s, say $\delta_s > 0$. Then, Assertion (ii) hods for $\delta := \min \delta_s$, for $s \in \underline{\mathbf{S}}$. $\square$

Assertion (iii) is immediate from the definition of equilibria, of the sets P_s, for each $s \in \underline{\mathbf{S}}$, and of Assertion (ii). $\square$

End of the proof: Lemmata 1 insures the desired contradiction with Claim 1-(v) (or the fact that the payoff and information structure, $[V, (S_i)]$, is arbitrage-free). $\square$

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